Lec. 23: Travelling Salesperson Problem (TSP)

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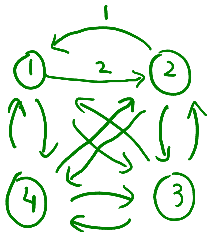
Weighted Cost Adjacency Matrix



	1	2	3	4
1	0	2	9	10
2	1	0	6	4
3	15	7	0	8
4	6	3	12	0

c_{ik}

Set of cities $1, 2, \ldots, n$

- Start from 1.
- Need to go through every city once and only once and return to 1.
- Need to find a route that does this in minimum cost.

S (2,3,4)

Solution

- Let $g(i, S)$ be the minimum cost of path
  1) starting from i
  2) going through every city in Set S once and only once.
  3) and ending at 1

S

$i \rightarrow$ 2→3→4, 2→4→3, 3→2→4, 3→4→2, 4→2→3, 4→3→2 $\rightarrow 1$

- From i, next city can be any city in S
  For city k, cost will be $c_{ik} + g(k, S\setminus\{k\})$

  c_{ik}: cost of going from i to k

  $g(k, S\setminus\{k\})$: min cost of going from k to 1 through $S\setminus\{k\}$ cities.

$S\setminus\{k\} = S - \{k\}$

$\setminus$ = set difference

- Best k should be one that gives min. cost

$$\underset{\text{larger set}}{g(i, S)} = \min_{k \in S} \; c_{ik} + \underbrace{g(k, S\setminus\{k\})}_{\text{recursive breakdown in terms of subproblems on smaller subsets } S\setminus\{k\}}$$

Base Case $g(i, \emptyset) = c_{i1}$ — starting from i, going through $\emptyset$, ending at 1

- $$g(1, \{2, 3, \ldots, n\}) = \min_{k \in \{2,3,\ldots,n\}} c_{1k} + g(k, \{2, \ldots, n\}\setminus\{k\})$$

- Best to solve in bottom-up fashion starting from

$$\left.\begin{array}{l} g(2, \emptyset) = c_{21} \\ g(3, \emptyset) = c_{31} \\ \vdots \\ g(n, \emptyset) = c_{n1} \end{array}\right\} n-1$$

Then

$$g(3,\{2\}) = c_{32} + g(2,\phi)$$
$$g(4,\{2\}) = c_{42} + g(2,\phi)$$
$$\vdots$$
$$g(n,\{2\}) = c_{n2} + g(2,\phi)$$
$$\underbrace{}_{n-2}$$

$$g(2,\{3\}) = c_{23} + g(3,\phi)$$
$$g(4,\{3\}) = c_{43} + g(3,\phi)$$
$$\vdots$$
$$g(n,\{3\}) = c_{n3} + g(3,\phi)$$
$$\underbrace{}_{n-2}$$

$$\cdots$$

$$g(2,\{n\}) = c_{2n} + g(n,\phi)$$
$$\vdots$$
$$g(n-1,\{n\}) = c_{n-1,n} + g(n,\phi)$$
$$\underbrace{}_{n-2}$$

$$\underbrace{}_{n-1}$$

$$(n-1)(n-2)$$

Then

$$g(i, \underbrace{\{a,b\}}_{\text{size } 2}) = \min_{k\in\{a,b\}} \left(c_{ik} + g(k, \underbrace{\{a,b\}\setminus k}_{})\right)$$

- size 1
- all known in bottom-up approach

All the way upto

$$\left.\begin{array}{l} g(2, \{3,4,\dots,n\}) \\ g(3, \{2,4,\dots,n\}) \\ \vdots \\ g(n, \{2,3,\dots,n-1\}) \end{array}\right\}$$

all known in bottom-up approach

and then

$$g(1,\{2,\dots,n\}) = \min_{k\in\{2,\dots,n\}} c_{1k} + g(k,\{2,\dots,n\}\setminus k)$$

- Time Complexity is $O(n^2 2^n)$, i.e exponential.
- No better solution is known. So, TSP qualifies as a "hard" problem.