CS-570 Computer Vision

Nazar Khan

Department of Computer Science
University of the Punjab

13. Transformations I: Affine and Projective

Homogenous Coordinates

- Vectors that we use normally are in *Cartesian coordinates* and reside in Cartesian space $\mathbb{R}^d$.
- Appending a 1 as the last element of a Cartesian vector yields a vector in *homogenous coordinates*.

$\mathbf{v}$	$\hat{\mathbf{v}}$
$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

- A homogenous vector resides in the so-called *projective space* $\mathbb{P}^d = \mathbb{R}^{d+1} \setminus \mathbf{0}$.
  - Projective space is just Cartesian space with an additional dimension *but* without an origin.
  - Dimensionality of $\mathbb{P}^d$ is $d + 1$.

Projective Space

- $\mathbb{R}^d$ to $\mathbb{P}^d$: Append by 1.
- $\mathbb{P}^d$ to $\mathbb{R}^d$: Divide by last element to make it 1 and then drop it.

$$\hat{\mathbf{v}} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} \longrightarrow \mathbf{v} = \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

- This means that in projective space, any vector $\mathbf{v}$ and its scaled version $k\mathbf{v}$ will *project down* to the same Cartesian vector.
- That is, $\mathbf{v}$ is *projectively equivalent* to $k\mathbf{v}$. Written as

$$\mathbf{v} \equiv k\mathbf{v} \tag{1}$$

for $k \neq 0$.

Affine Transformation in $\mathbb{P}^2$

- Consider the following linear transformation from $\mathbb{P}^2$ to $\mathbb{P}^2$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Note that the last component will remain unchanged.
- Every affine transformation is invertible.
- Six degrees of freedom (DoF).
- An affine transformation matrix can perform 2D rotation, scaling, shear or translation.
- Any sequence of affine transformations is still affine (look at the last row).

Affine Transformation

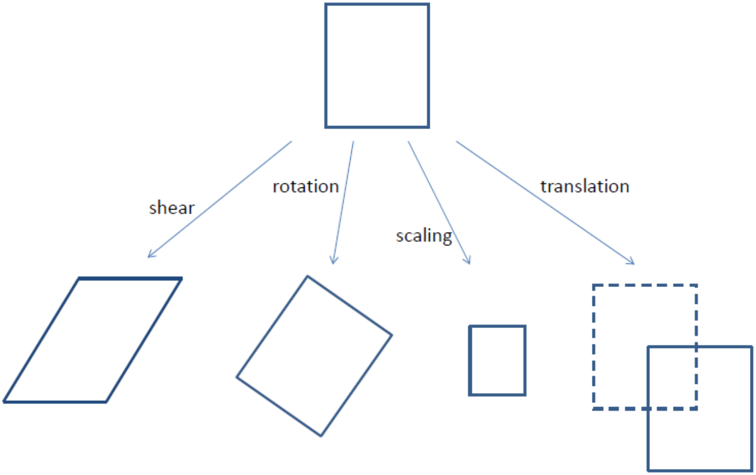


Figure: Capabilities of an affine transformation matrix.

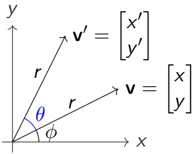
Affine Transformation

Scaling	Shear	Translation	Rotation
$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & sh_x & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$x' = s_x x$	$x' = x + sh_x y$	$x' = x + t_x$	$x' = x\cos\theta - y\sin\theta$
$y' = s_y y$	$y' = y + sh_y x$	$y' = y + t_y$	$y' = x\sin\theta + y\cos\theta$

Note that translation cannot be written in matrix-vector form in Cartesian space.

Rotation Matrix
Derivation

For counter-clockwise rotation of **v** *around origin* by θ



$$\begin{aligned} x' &= r\cos(\phi+\theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ &= x\cos\theta - y\sin\theta \\ y' &= r\sin(\phi+\theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta \\ &= x\sin\theta + y\cos\theta \end{aligned}$$

Therefore

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{2}$$

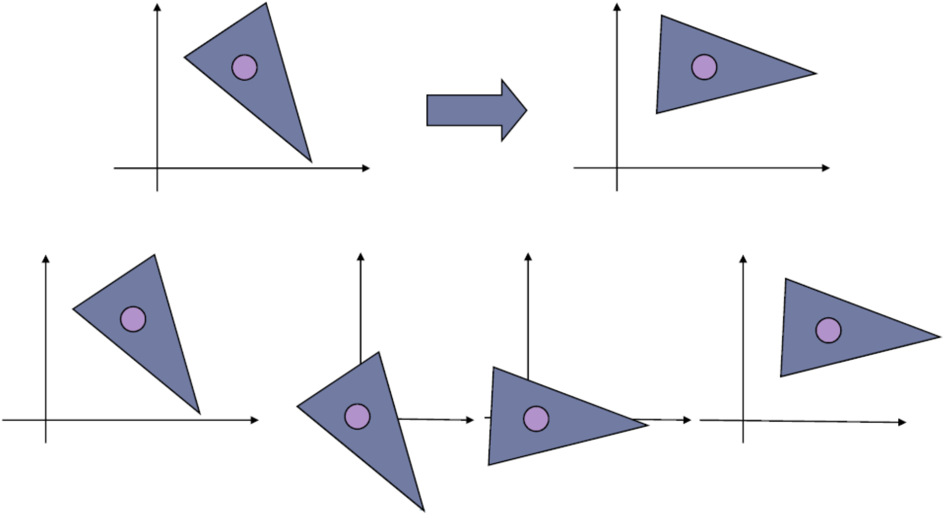
Rotation Matrix
Properties

- For any rotation matrix $\mathbf{R}$
  1. Each row is orthogonal to the other. Same for columns.
  2. Each row has unit norm. Same for columns.
- Such matrices are called *orthonormal* matrices.

$$\mathbf{R}^T\mathbf{R} = \mathbf{I}$$

- They preserve length of the vector being transformed.

Rotation around an arbitrary point



Order matters!

Rotation/scaling/shear followed by translation

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & sh_x & 0 \\ sh_y & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & sh_x & t_x \\ sh_y & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

is not the same as translation followed by rotation/scaling/shear.

$$\begin{bmatrix} s_x & sh_x & 0 \\ sh_y & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & sh_x & s_x t_x + sh_x t_y \\ sh_y & s_y & sh_y t_x + s_y t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Projective Transformation

- Last row of affine transformation matrix is always $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.
- If this condition is relaxed we obtain the so-called *projective transformation*.

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$$

- Also called *homography* or *collineation* since lines are mapped to lines.

Projective Transformation

- Linear in $\mathbb{P}^2$ but non-linear in $\mathbb{R}^2$ because 3rd coordinate of $\mathbf{v}'$ is not guaranteed to be 1.

$$\begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} h_1 x + h_2 y + h_3 \\ h_4 x + h_5 y + h_6 \\ h_7 x + h_8 y + h_9 \end{bmatrix} \implies \begin{array}{l} x' = \frac{h_1 x + h_2 y + h_3}{h_7 x + h_8 y + h_9} \\ y' = \frac{h_4 x + h_5 y + h_6}{h_7 x + h_8 y + h_9} \end{array}$$

- The 3rd coordinate is now a function of the inputs x and y and division involving them makes the transformation non-linear.

Projective Transformation
Degrees of Freedom

- Projective transformation has only 8 degrees of freedom.
  - In projective space, $\mathbf{v} \equiv k(\mathbf{v})$ for all $k \neq 0$ because both correspond to the same point in Cartesian space. So

$$k(\mathbf{v}) \equiv \mathbf{v} \implies k(\mathbf{Hv}) \equiv \mathbf{Hv} \implies k\mathbf{Hv} \equiv \mathbf{Hv} \implies k\mathbf{H} \equiv \mathbf{H}$$

  - Let $\mathbf{H}' = \frac{1}{h_9}\mathbf{H}$. Clearly, $h'_9 = 1$ and therefore $\mathbf{H}'$ has 8 free parameters.
  - But since $\mathbf{H}' \equiv \mathbf{H}$, $\mathbf{H}$ must also have only 8 free parameters.