SE 461 Computer Vision

Nazar Khan
PUCIT

Lectures 5, 6 and 7

Disclaimer

- Any unreferenced image is taken from the following web-page
  - http://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/

Note

- If a hammer is the only tool you have, you will look at every problem as a nail.
- The more tools you have, the more problems you can tackle.
- Our foray into the “Fourier world” is an attempt to gather as many tools as we can.

Fourier Transform

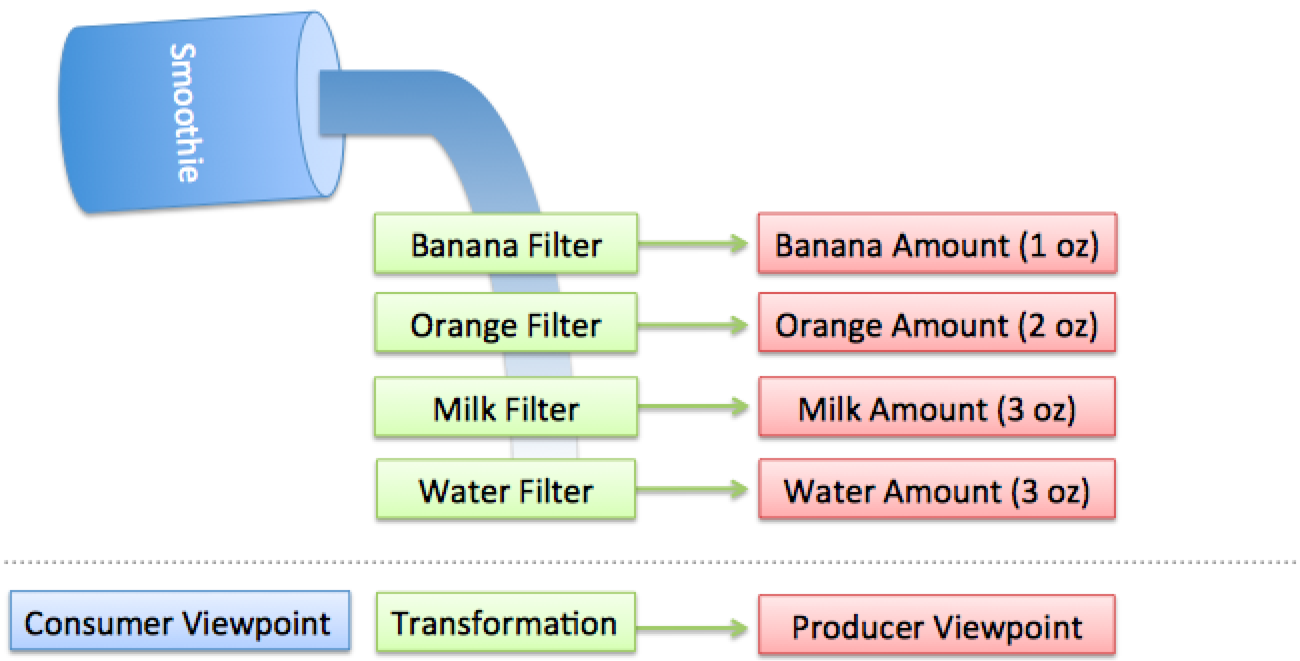
- One of the deepest mathematical insights.
- For any signal, it extracts its “ingredients”.
  - This is a **very powerful** idea.
  - Given an observation, it gives you the causes.
  - Given an image, it gives you its constituents.
- Understanding the Fourier Transform requires some of the **most beautiful mathematics** ever invented.

Fourier Transform

- The mathematics can become (more than) a little bit overwhelming.
- So we’ll break it down into smaller, easier steps.

Fourier Transform – An Analogy

Smoothie to Recipe



Source: http://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/

Fourier Transform

- We start with some pre-requisite mathematics.
  - Remember, math is not magic!
  - You **can** understand it if you take the correct perspective.

Mathematical Background

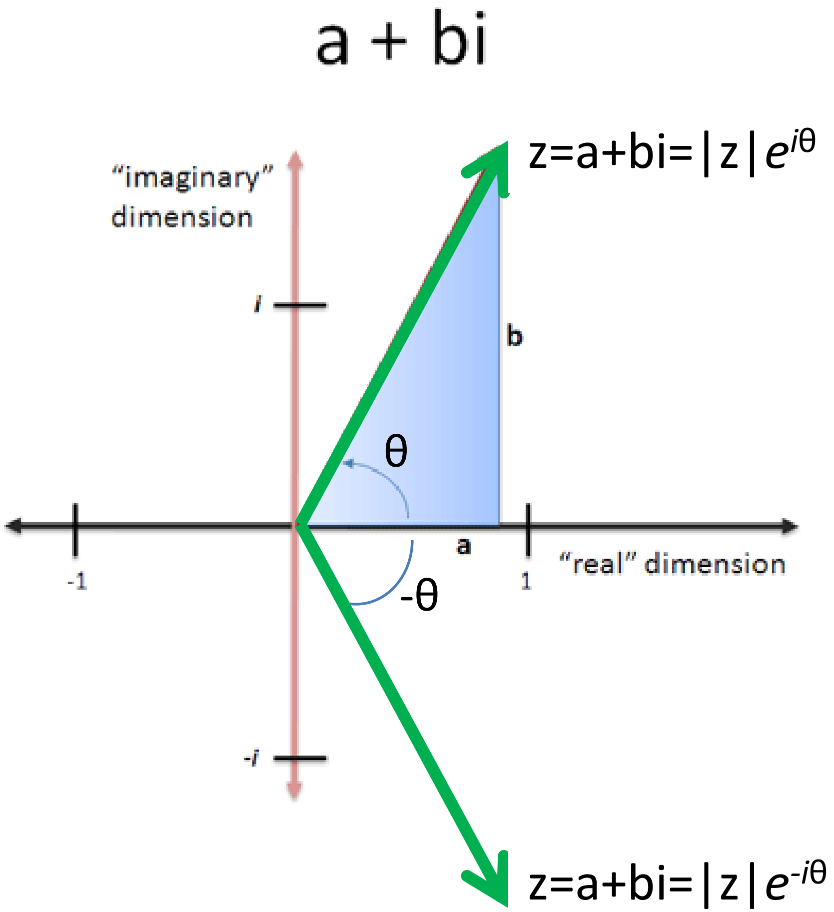
- π
  - circumference/diameter of **any** circle.
  - universal constant (π = 3.14159265...)
- e
  - Euler's number (e = 2.71828182...)
- i
  - non-existent, imaginary number (what!!!!)
  - makes analysis and computations easier ($i^2=-1$)

Complex Numbers

- Real numbers are represented by R^1.
- We can write any real number x as x+0*i.*
- Therefore, R^1 is contained in space of complex numbers C^1.
  - Complex numbers z have a real part Re(z) and an imaginary part Im(z).
- Basis vector for R^1 is the scalar 1.
- Basis vectors for C^1 are {(1,0),(0,*i*)}.

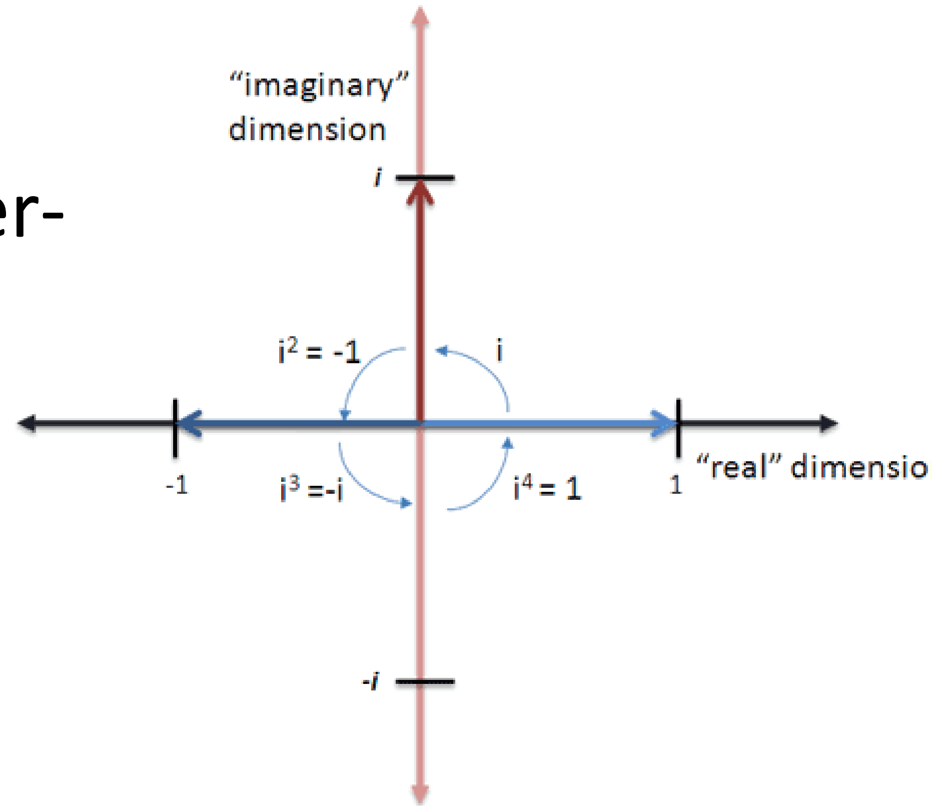
Complex Numbers

- Norm (magnitude, modulus) of z is given by $|z|=\mathrm{sqrt}(a^2+b^2)$.
- Phase is the angle $\theta=\arctan(b/a)$.
- A complex number can also be represented in Polar form $z=a+bi=|z|e^{i\theta}$.
- Conjugate of z is given by $\mathrm{conj}(z)=a-bi=|z|e^{-i\theta}$.
- **HW: Compute the values of sqrt(z*z) and sqrt(z*conj(z)). Which one yields the norm of z?**



Multiplication by *i* Represents 90° Rotation in *C*

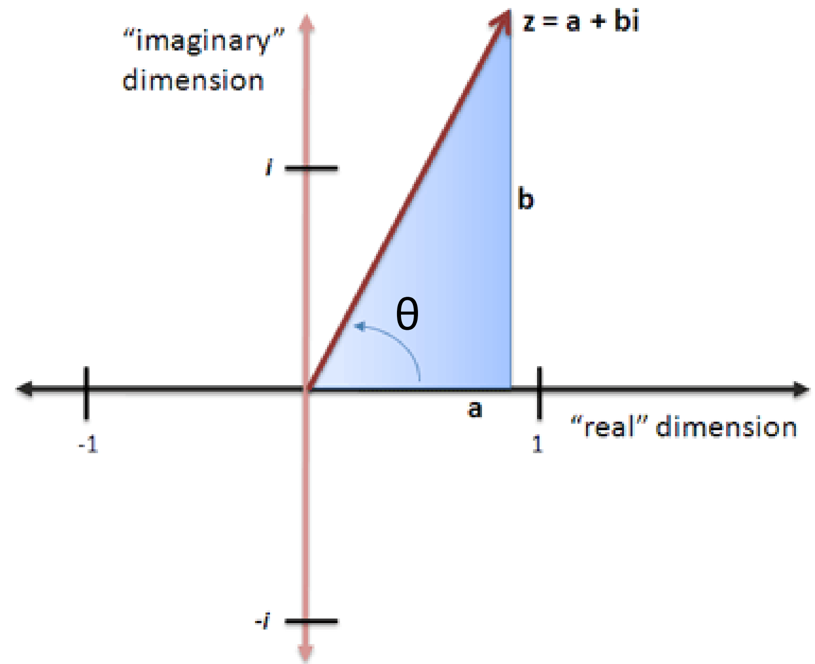
- Multiplication by *i* is a **rotation** by 90° counter-clockwise in *C*.
  - 1**i*=*i*
  - 1**i***i*=-1
  - 1**i***i***i*=-*i*
  - 1**i***i***i***i*=1



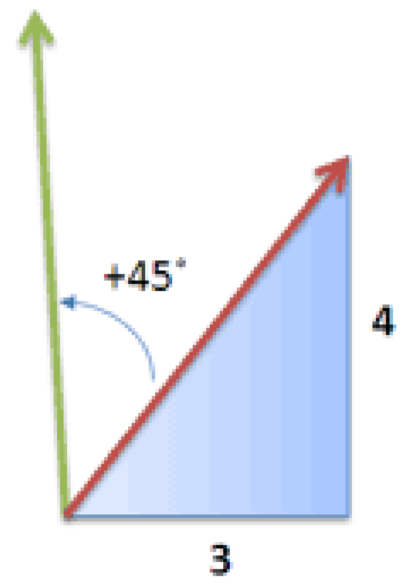
Multiplication by Complex Number Represents Rotation in *C*

- Multiplication by any complex number z=a+b*i* causes **rotation** by its **angle** θ=arctan(b/a)

a + bi



Find the heading

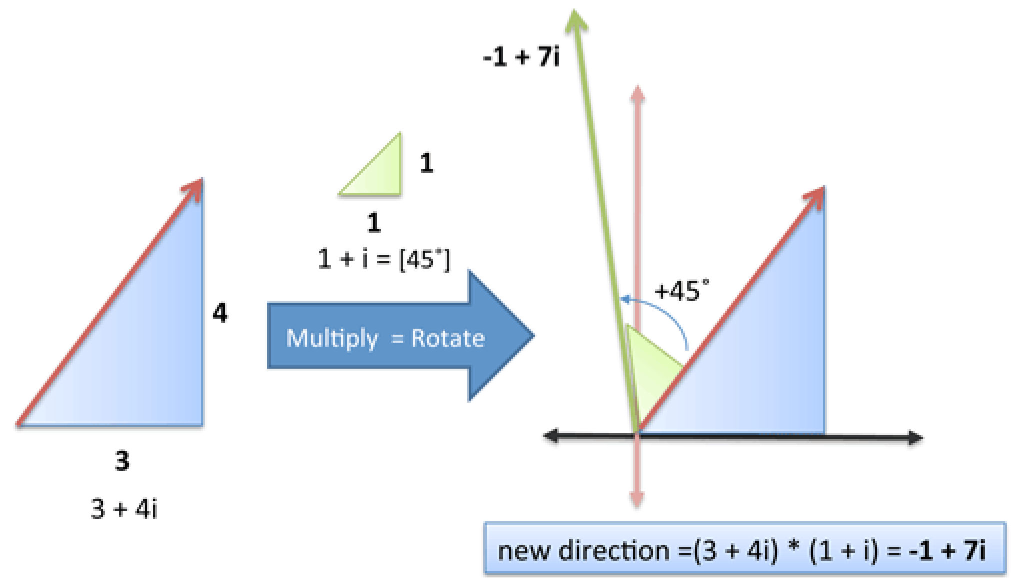


- Suppose I'm on a boat, with a heading of 3 units East for every 4 units North. I want to change my heading 45 degrees counter-clockwise. What's the new heading?
- The usual method: arctan (4/3)+45 = 98.13°

Exploiting the Complex Space

- Represent the original direction in the complex plane where **rotation=multiplication**
  - 3+4i
- Find the complex number representing 45° rotation.
  - 1+1i
  - Angle = arctan(1/1)=45°
- Multiply the two complex numbers.
- New direction is -1 unit East, 7 units North.
  - A complex number -1+7i with angle=arctan(7/-1)=98.13° as before

Applying Complex Numbers



$$
\begin{aligned}
(3+4i)\cdot(1+i) &= 3+3i+4i+4i^2 \\
&= 3+7i \qquad +4(-1) \\
&= -1+7i
\end{aligned}
$$

The Bigger Picture

- The complex space C is just a generalization of the real space R where rotation amounts to multiplication.
- We don't care about C itself but we care about the fact that in C complicated rotations can be represented as simply as multiplications.
  - We don't care whether –ve numbers actually exist or not, we care that they make calculations of profit/loss or debit/credit easier.

Euler

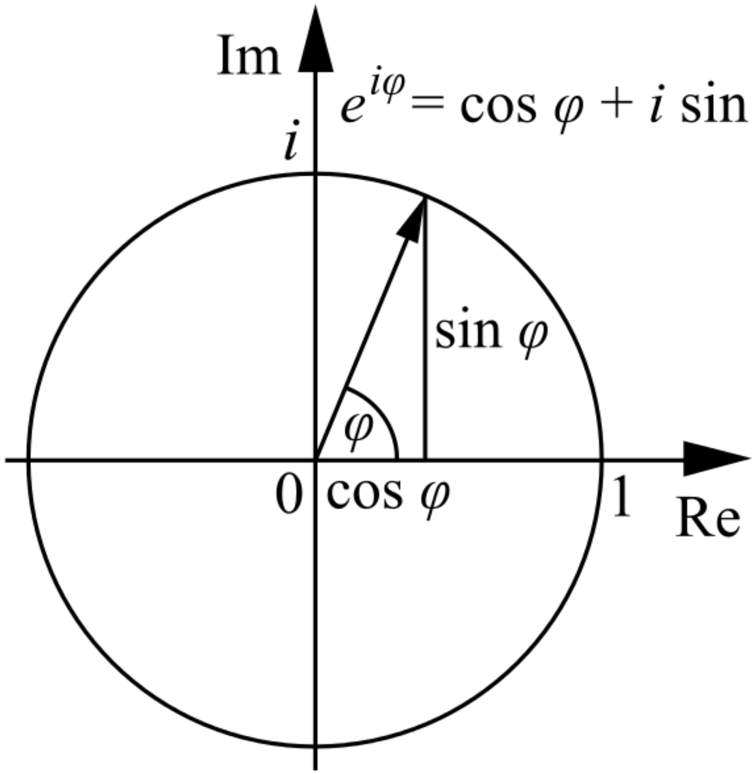
- One of the greatest mathematicians ever.
- Fundamental contributions in calculus, graph theory, optics, fluid dynamics, mechanics, astronomy and even music theory.
- Almost totally blind for the last 20 years of his life.
  – Yet did the most productive work during this time.

Source: http://en.wikipedia.org/wiki/Leonhard_Euler

Euler's Formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

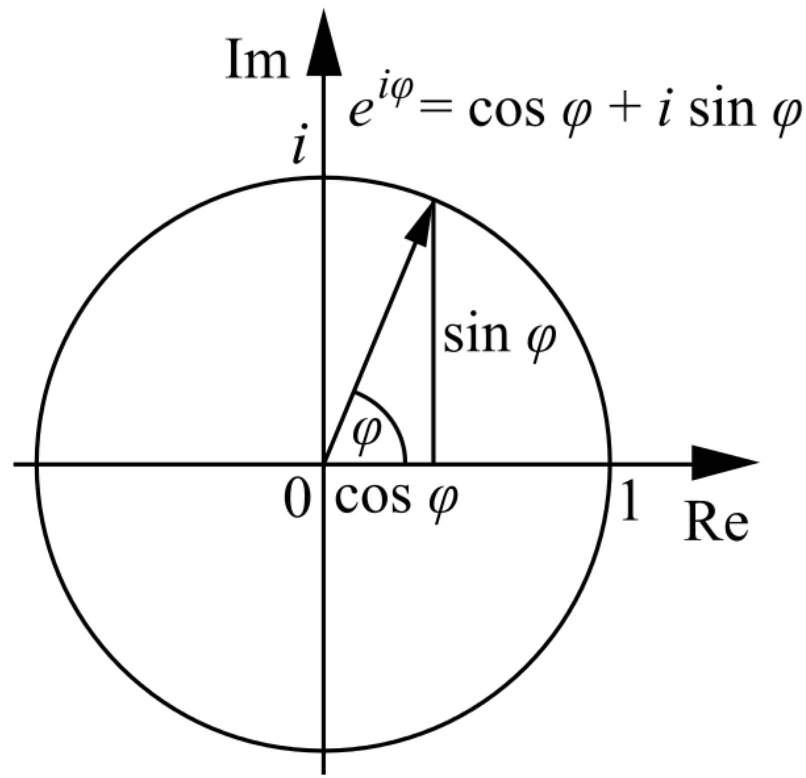
- **Mathematics does not get more beautiful than this equation.**
- What you can describe using sinusoids, you can describe using the numbers *e*=2.71828182... and *i*=sqrt(-1)



In 1988, readers of the *Mathematical Intelligencer* voted it "the Most Beautiful Mathematical Formula Ever". In total, Euler was responsible for three of the top five formulae in that poll.

Euler's Formula

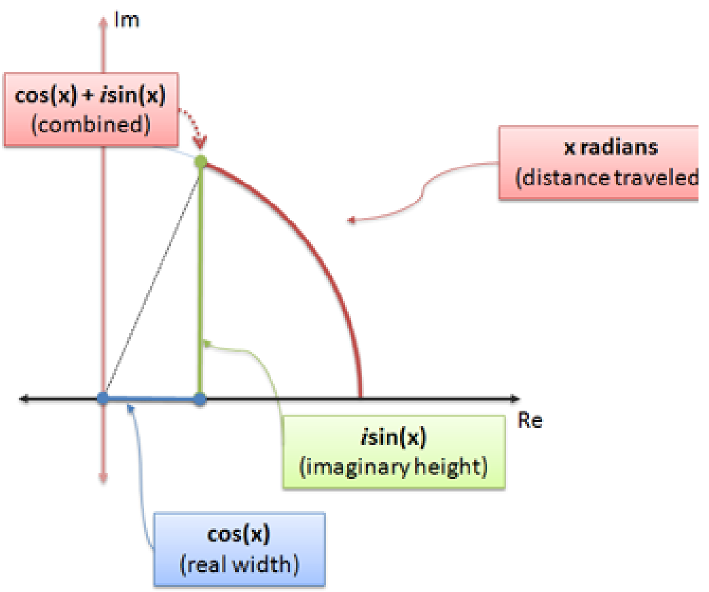
- What can we describe using *cos*(θ) and *sin*(θ)?
  - Positions on a circle.
- The formula says that that position is $2.7182818284^{\theta\sqrt{-1}}$ or simply $e^{i\theta}$.



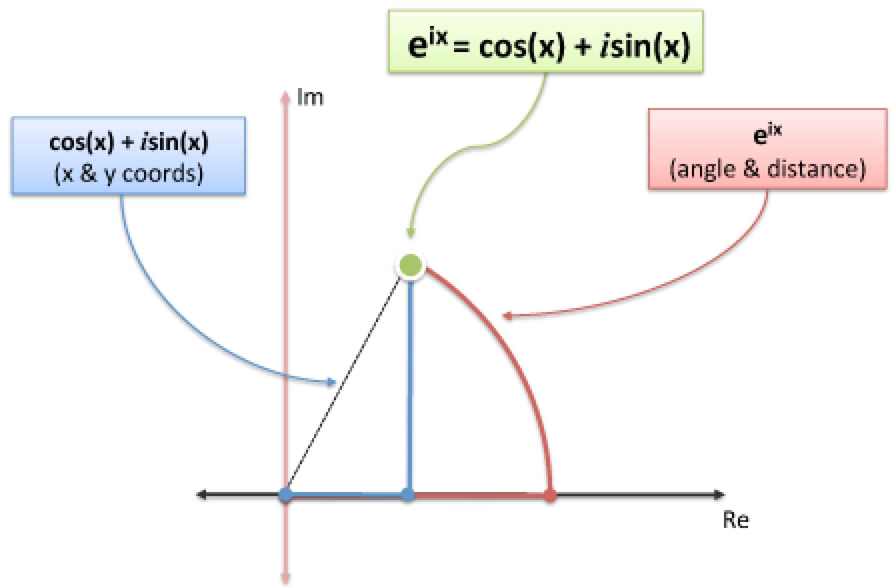

```
In Matlab:
>> [exp(sqrt(-1)*pi/4); cos(pi/4)+i*sin(pi/4)]
ans =
   0.7071 + 0.7071i
   0.7071 + 0.7071i
```

Euler’s Formula

Traversing A Circle



Two Paths, Same Result



Euler's Formula – The Bigger Picture

- **Describes circular motion.**
- Two ways to describe motion
  - Cartesian: Go 3 units east and 4 units north
  - Polar: Go 5 units at an angle of 71.56 degrees
- Depending on the problem, polar or Cartesian coordinates are more useful.
- **Euler's formula lets us convert between polar and Cartesian representation to use the best tool for the job.**

The link between Euler’s Formula and the Fourier Transform

- **Fourier’s claim: Any signal can be made from circular motion.**
- Euler's formula generates <u>all</u> circular motions.
- So Euler’s formula is the tool that the Fourier Transform needs to decompose signals into circular motions.

Fourier Transform

- Fourier Transform factorises the <u>angular distance θ</u> into <u>angular speed ω</u> and <u>time t</u>.
  - θ is angular distance along the circle (0—2π).
- Since $\theta = \omega t$, we can write $e^{i\theta} = e^{i\omega t}$
  - So $e^{i\omega t}$ determines how far we have moved along the circle in time t travelling at speed ω.
- By varying ω and t, we can compute how far a circular motion with speed ω will be at time t.

Fourier Transform

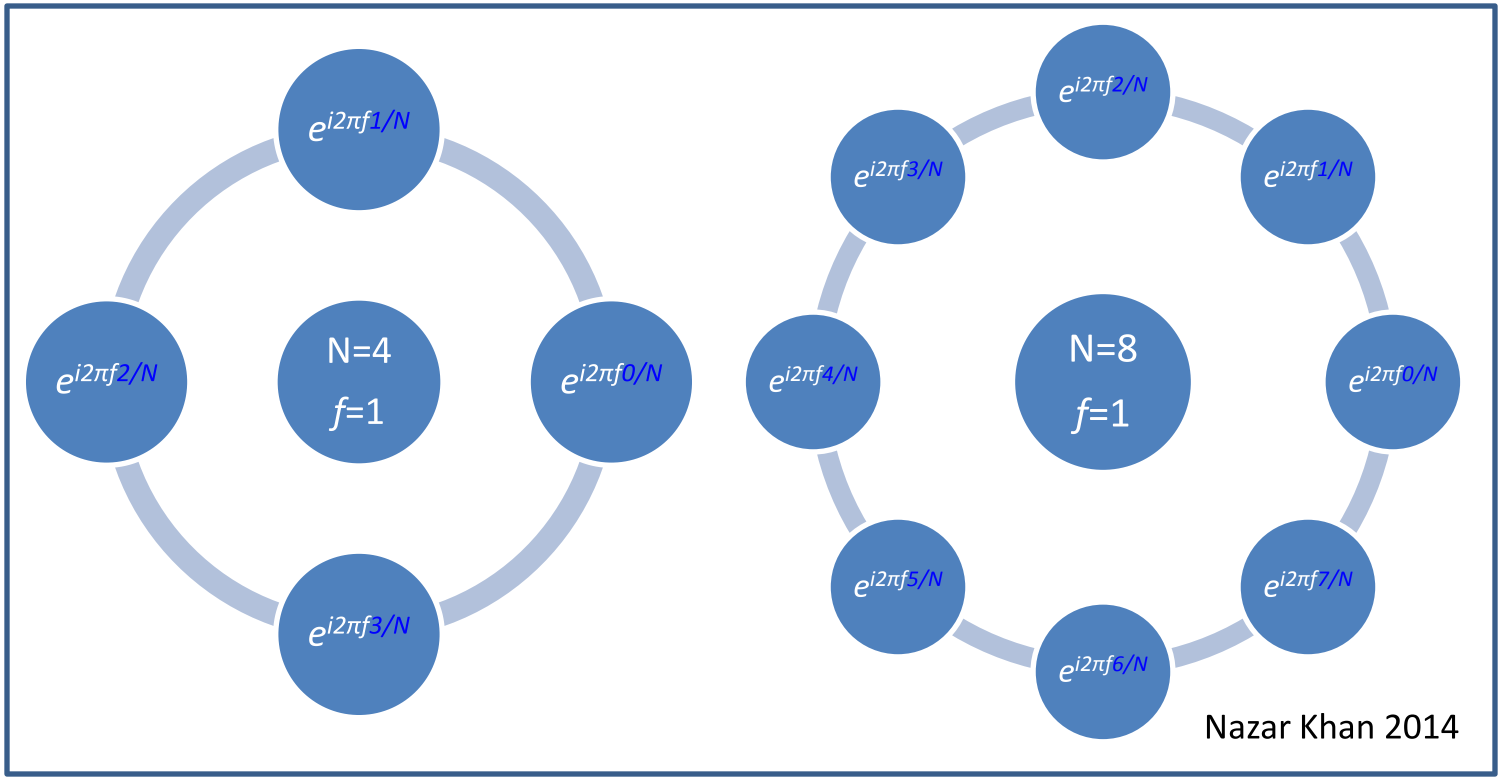
- Angular speed $\omega = 2\pi f$ where f is the frequency in cycles per unit time. (**HW: Verify this. Hint: Just look at the definitions and/or units of ω and f.**)
- So we can write $e^{i\theta} = e^{i\omega t} = e^{i2\pi ft}$
  - So $e^{i2\pi ft}$ determines how far we have moved along the circle in time t travelling with a frequency f.
- **By varying f and t, we can compute how far a circular motion with frequency f will be at time t.**

Fourier Transform

- Let total time be 1 second.
- Assume $x_0, x_1, \ldots, x_{N-1}$ are signal values in a time of 1 second.
- Value x_n occurs at time t=n/N seconds.
- Position on the circle at time t=n/N is given by $e^{i\theta} = e^{i\omega t} = e^{i2\pi f t} = e^{i2\pi f n/N}$
- This gives us N positions along a circular motion with frequency *f*.
- The signal is also N dimensional.
- Project signal onto the circular motion by taking the dot product.

Fourier Transform

- Position on the circle at time t=n/N is given by $e^{i\theta} = e^{i\omega t} = e^{i2\pi ft} = e^{i2\pi fn/N}$
- This gives us N positions along a circular motion with frequency f.
- Project signal onto the circular motion by taking the dot product.



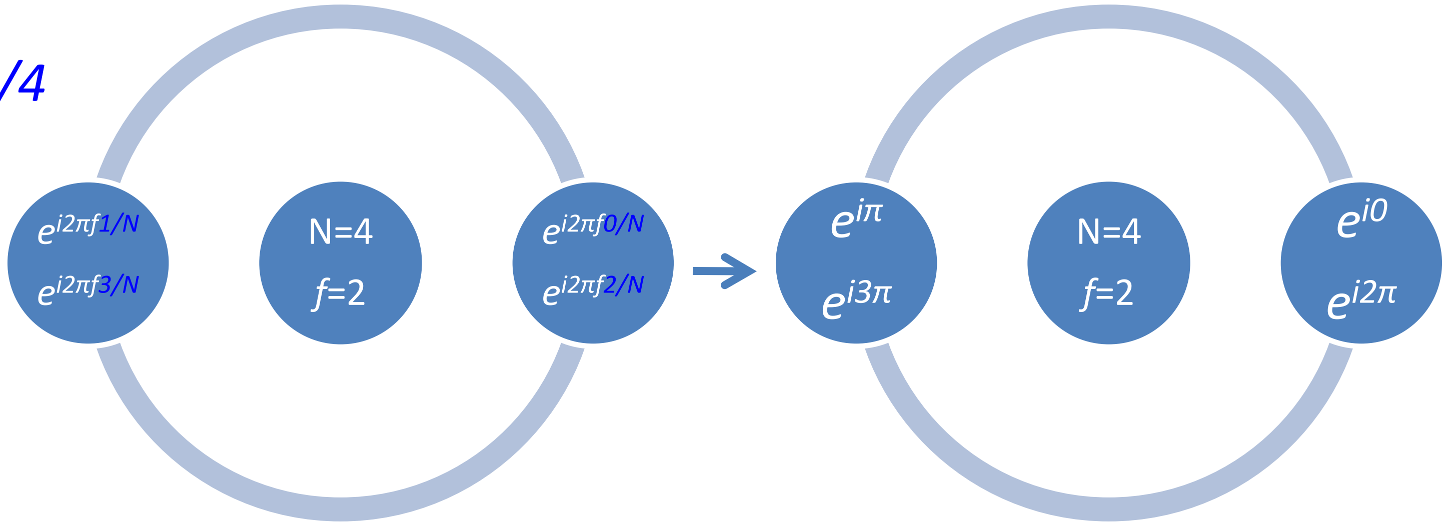
Fourier Transform

- f=2 implies 2 cycles/second.
- $e^{i2\pi f3/N}$

  $= e^{i2\pi 2*3/4}$

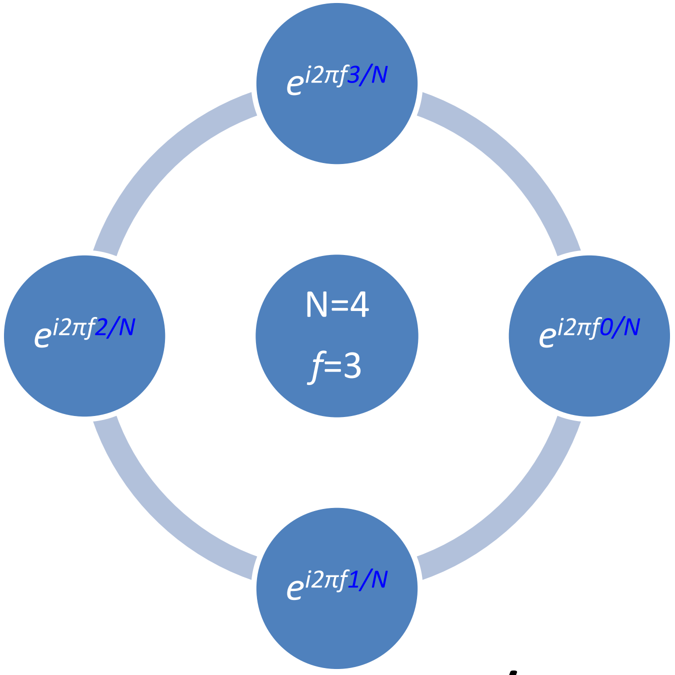
  $= e^{i3\pi}$

  $= -1+0i$



Fourier Transform

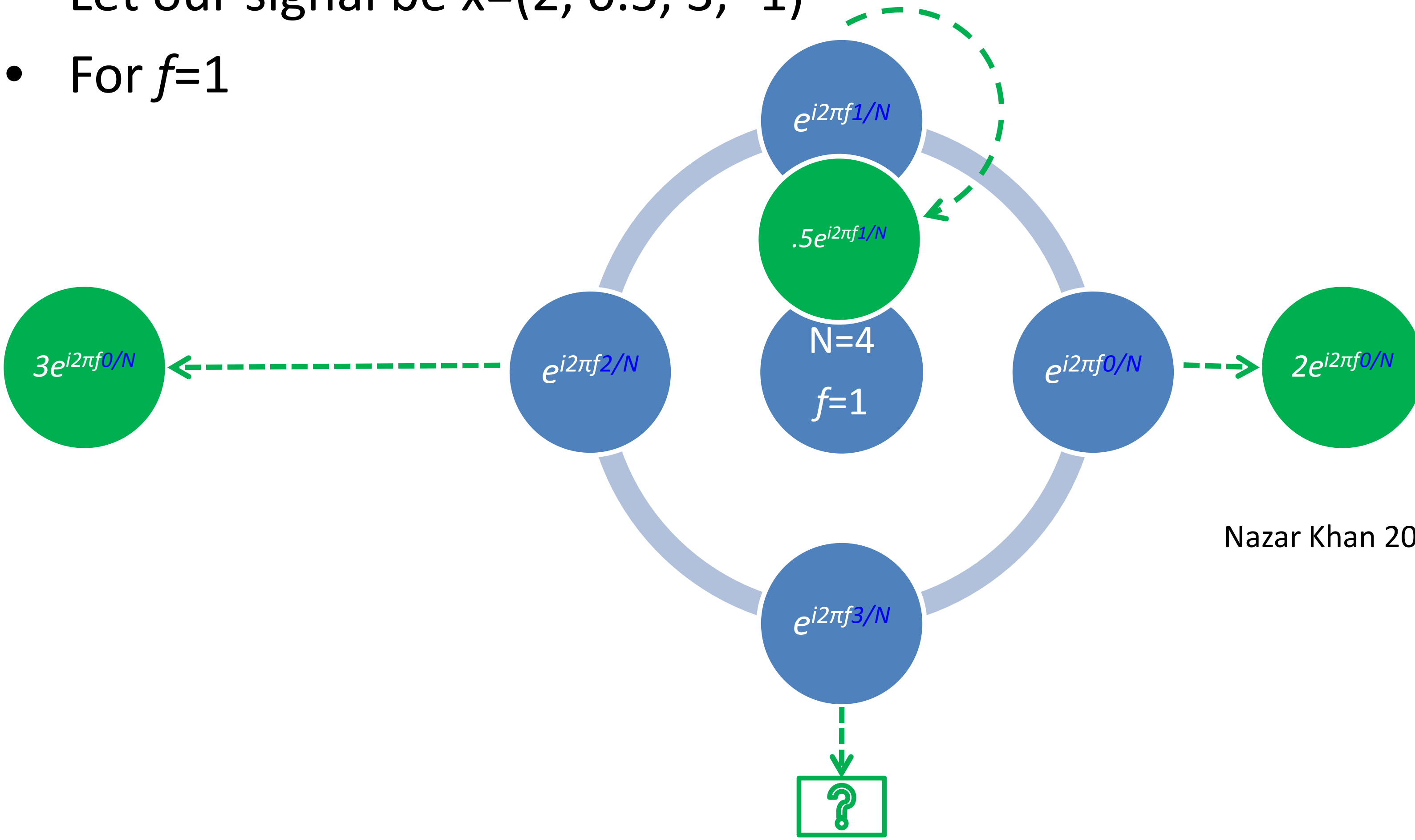
- f=3



- Notice the positions at t=1/N and 3/N.

Fourier Transform

- **Scale the n/N^{th} position by the signal value x_n.**
- Let our signal be x=(2, 0.5, 3, -1)
- For f=1



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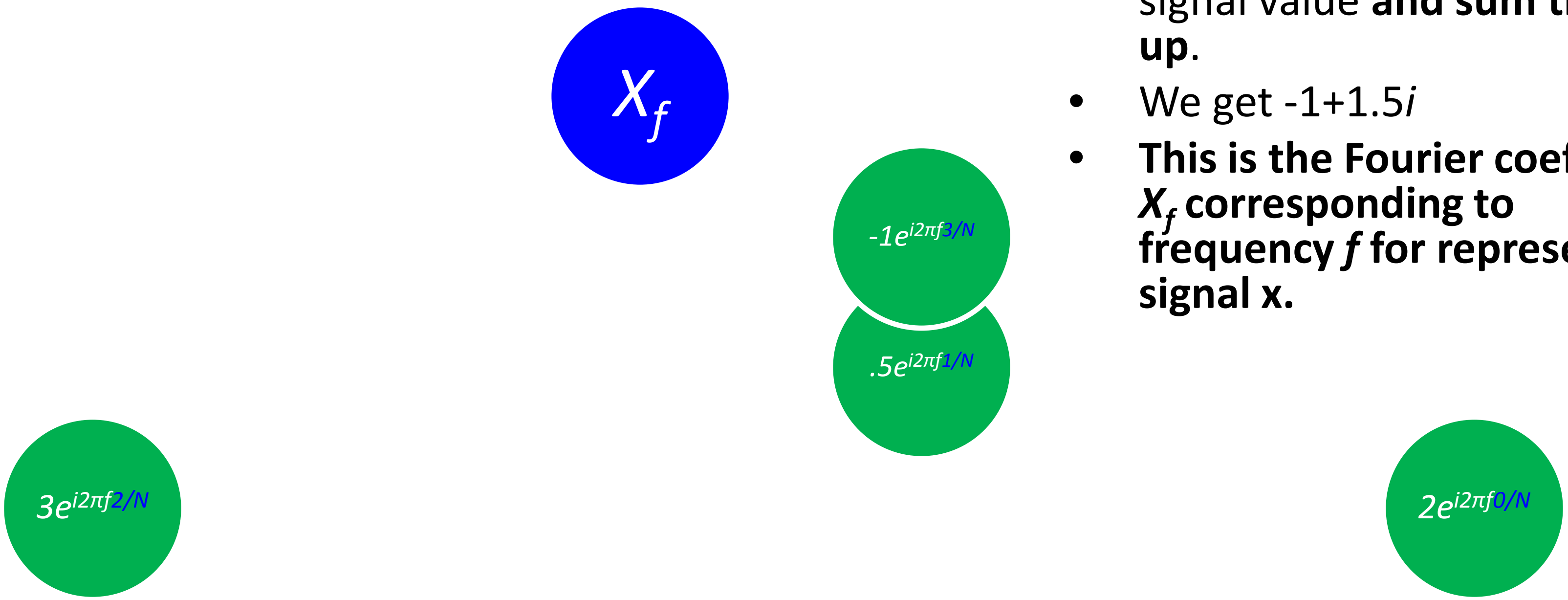
Fourier Transform



- Scale each position by the signal value **and sum them up**.
- We get -1+1.5*i*
- **This is the Fourier coefficient *X_f* corresponding to frequency *f* for representing signal x.**

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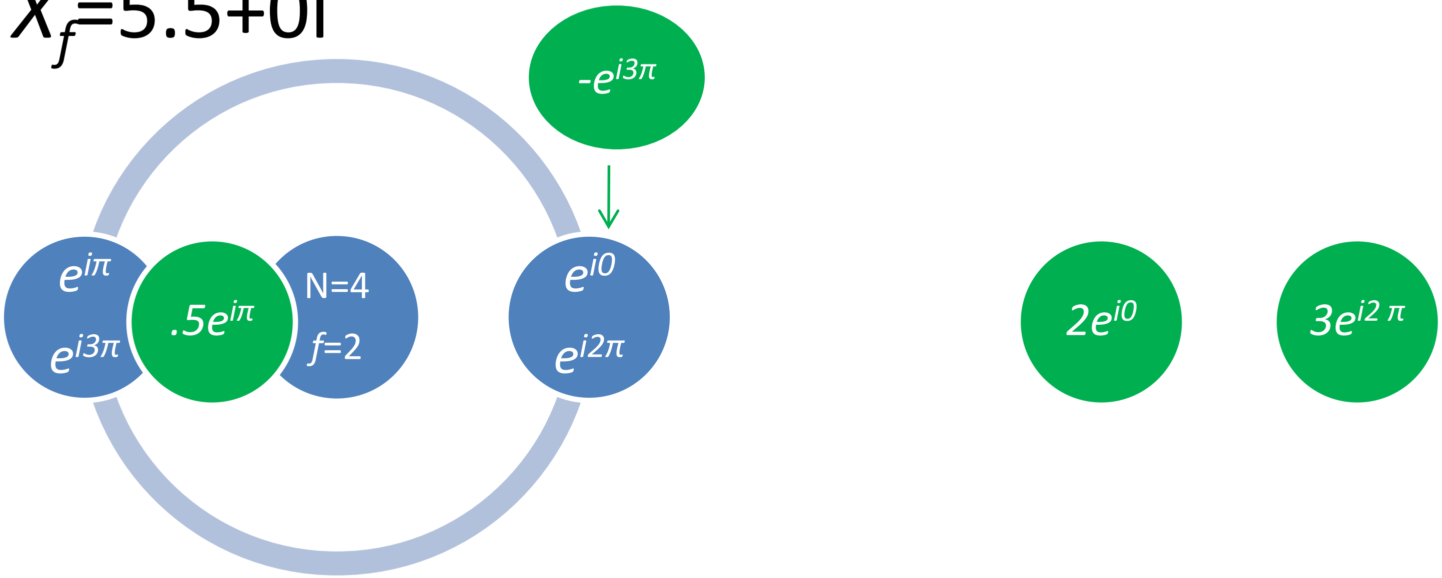
Fourier Transform

- $f=2$
- $X_f=5.5+0i$



- **H.W. Find X_f for $f=3$.**

Fourier Transform – The Bigger Picture

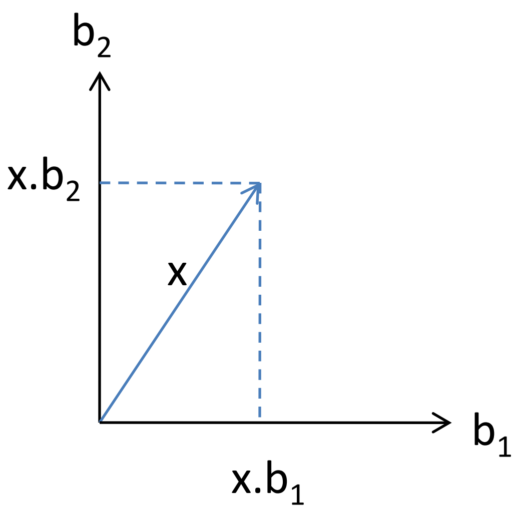
- For any circular path with frequency *f*
  - For every time instant n=0 to N-1
    - Multiply $x_n . e^{i2\pi f n/N}$
  - Add the products.
  - That is, compute the inner product

$$x \cdot e^{i2\pi f} = \sum_{n=0}^{N-1} x_n e^{i2\pi f n/N}$$

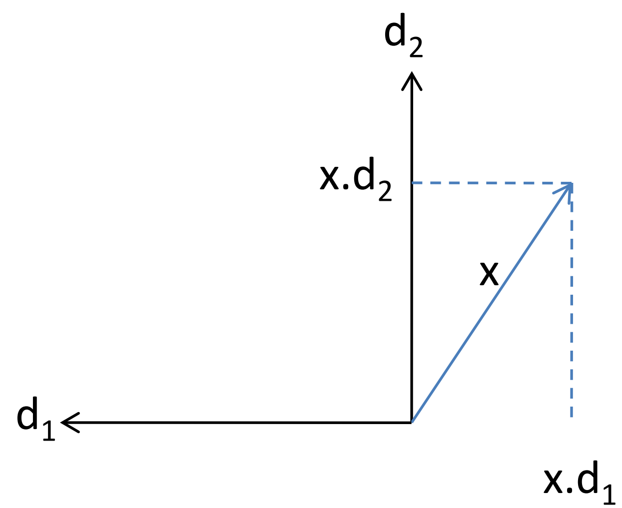
- **For every frequency *f*, we <u>project</u> the signal x onto the circular motion basis $e^{i2\pi f}$.**

Projection

- A 2D vector x can be represented in an orthonormal basis $\{b_1,b_2\}$ by the formula $x=(x.b_1)b_1+(x.b_2)b_2$.
  - Coefficient for basis vector k is the <u>projection</u> $(x.b_k)$.



$x=(x.b_1)b_1+(x.b_2)b_2$



$x=(x.d_1)d_1+(x.d_2)d_2$

Fourier Transform – Projection onto Circular Motion

- For the Fourier transform, the N dimensional signal vector x is **projected** onto the circular basis vectors $\boldsymbol{e^{i2\pi f}}$.
  - Coefficient for basis vector with frequency *f* is the projection (x.$\boldsymbol{e^{-i2\pi f}}$).

$$X_f = x \cdot e^{-i2\pi f}$$

$$= x_0 e^{-i2\pi f \, 0/N} + \cdots + x_{N-1} e^{-i2\pi f \frac{N-1}{N}}$$

  - Do you notice something strange in the projection?

Fourier Transform – Projection onto Circular Motion

- Why the negative sign in the exponent?
- In order to measure lengths in any number space, a norm must be defined such that |x| = sqrt(x.x) = length of vector x.
- In the space of Complex numbers, inner product is defined as x.y = x*conj(y) where conj(y) = Re(y)-Im(y)i = |y|$e^{-i\theta}$.
- **HW: For a complex <u>vector</u> f=(f_1,…,f_N), compute f*f and f*conj(f). Which one yields the squared norm of f (given by $|f|^2=|f_1|^2+\ldots+|f_N|^2$)?**
- <u>The negative sign signifies conjugation of $e^{i2\pi f}$. So that the norm can be properly defined in Complex space.</u>

Fourier Transform

$$X_f = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n e^{-i2\pi f \, n/N} = \frac{1}{\sqrt{N}} x \cdot e^{-i2\pi f}$$

Decompose the signal into its constituent frequencies.

Inverse Fourier Transform

$$x_n = \frac{1}{\sqrt{N}} \sum_{f=0}^{N-1} X_f e^{i2\pi f \, n/N}$$

<u>Synthesize</u> the signal <u>from</u> its constituent frequencies.

Orthonormality of the Fourier Basis

- The basis vector for different frequencies *f* are orthonormal.
- Orthogonality

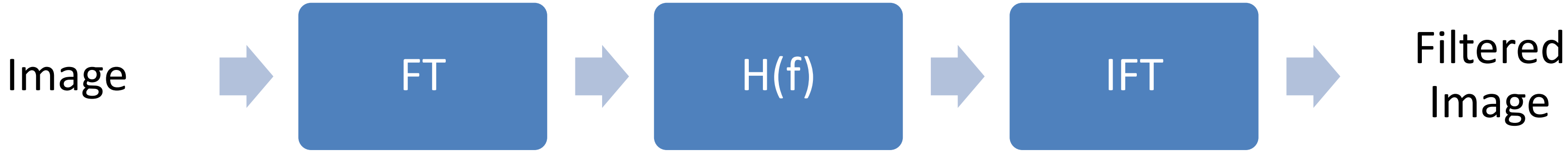
$$\frac{1}{\sqrt{N}}\left(e^{-i2\pi f_p {}^1\!/\!_N},\cdots,e^{-i2\pi f_p \frac{N-1}{N}}\right)\bullet\frac{1}{\sqrt{N}}\left(e^{-i2\pi f_q {}^1\!/\!_N},\cdots,e^{-i2\pi f_q \frac{N-1}{N}}\right)=0 \quad \forall p \neq q$$

- Normality

$$\frac{1}{\sqrt{N}}\left(e^{-i2\pi f_p {}^1\!/\!_N},\cdots,e^{-i2\pi f_p \frac{N-1}{N}}\right)\bullet\frac{1}{\sqrt{N}}\left(e^{-i2\pi f_q {}^1\!/\!_N},\cdots,e^{-i2\pi f_q \frac{N-1}{N}}\right)=1 \quad \forall p = q$$

- So the different frequencies do not interfere with each other in representing the signal.
- **HW: Prove orthonormality of Fourier basis.**

Frequency Domain Filtering Pipeline



Frequency Domain Low-Pass Filtering (Smoothing)

a b c

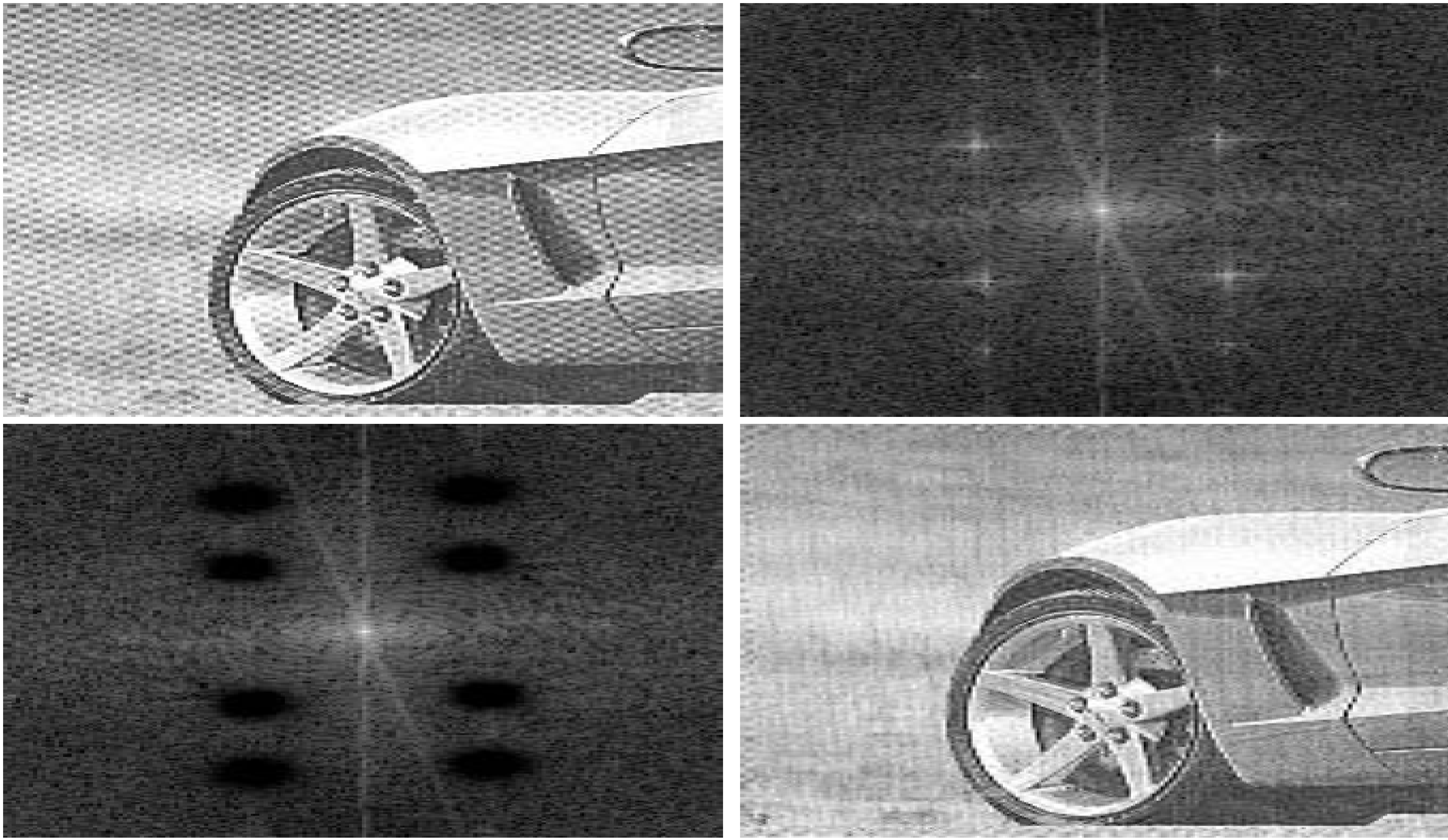
FIGURE 4.50 (a) Original image (784 × 732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

Source: Gonzalez & Woods

Frequency Domain High-Pass Filtering (Sharpening)

Source: Gonzalez & Woods

Frequency Domain Band-Pass Filtering

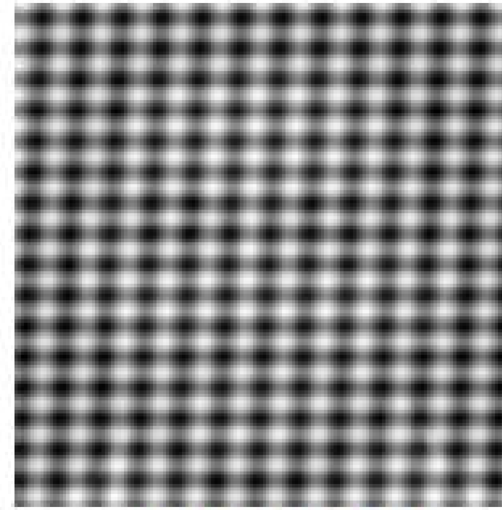


Source: Gonzalez & Woods

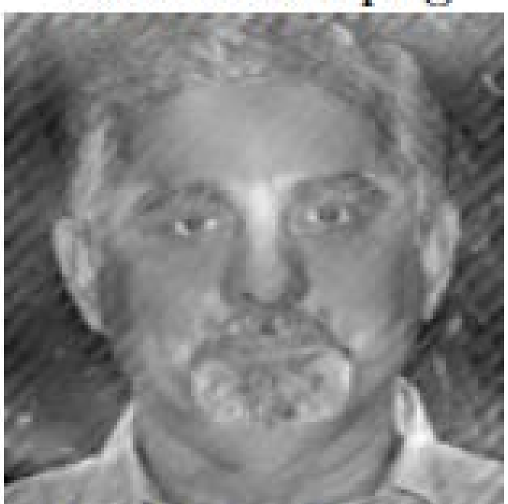
Frequency Domain Band-Pass Filtering

bossJailed1.png

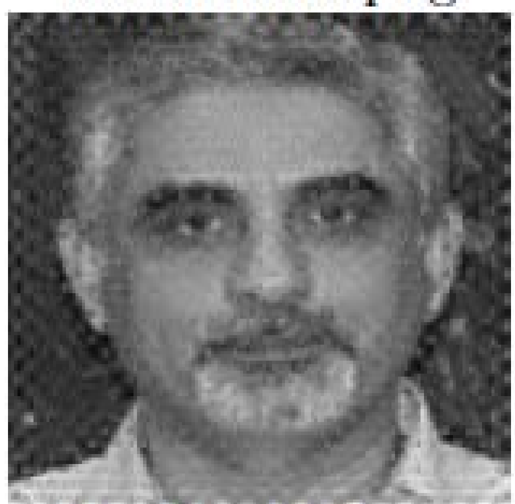
bossJailed2.png



mystery.png



bossFreed1.png



bossFreed2.png

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