CS 565 Computer Vision – Assignment 6 Hints

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1. **Homogeneous Coordinates**

   For a point $\boldsymbol{x} = (x_1, x_2)^T$ in Euclidean coordinates, the point $\tilde{\boldsymbol{x}} = (x_1, x_2, 1)^T$ is its counterpart in homogeneous coordinates. For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ or $\mathbb{P}^2$, the operator $\times$ denotes the cross product

   $$\boldsymbol{x} \times \boldsymbol{y} := \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$

   The cross product $\boldsymbol{x} \times \boldsymbol{y}$ is another vector that is orthogonal to both $\boldsymbol{x}$ and $\boldsymbol{y}$.

   Let $\boldsymbol{m}_1 = (x_1, y_1)^T$ and $\boldsymbol{m}_2 = (x_2, y_2)^T$ be two different points (i.e. $\boldsymbol{m}_1 \neq \boldsymbol{m}_2$) in Euclidean coordinates, and let $\boldsymbol{\ell}_1 = (a_1, b_1, c_1)^T$ and $\boldsymbol{\ell}_2 = (a_2, b_2, c_2)^T$ be non-parallel lines (e.g. $\boldsymbol{\ell}_1$ is defined by $a_1 x + b_1 y + c_1 = 0$).

   (a) Show that $\boldsymbol{m}_1$ lies on $\boldsymbol{\ell}_1$ if and only if $\tilde{\boldsymbol{m}}_1^T \boldsymbol{\ell}_1 = 0$.
   Prove the **if** part by writing $\tilde{\boldsymbol{m}}_1^T \boldsymbol{\ell}_1 = 0$ and showing that it leads to $a_1 x_1 + b_1 x_2 + c_1 = 0$.
   Prove the **only if** part by writing $a_1 x_1 + b_1 x_2 + c_1 = 0$ and showing that it leads to $\tilde{\boldsymbol{m}}_1^T \boldsymbol{\ell}_1 = 0$.

   (b) Let $\boldsymbol{m}_1$ be the intersection of $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$. Show that $\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2 = \tilde{\boldsymbol{m}}_1$ holds.
   Since $\boldsymbol{m}_1$ lies on $\boldsymbol{\ell}_1$, $\tilde{\boldsymbol{m}}_1^T \boldsymbol{\ell}_1 = 0$ which means that vector $\tilde{\boldsymbol{m}}_1$ is orthogonal to vector $\boldsymbol{\ell}_1$. Do a similar analysis for $\boldsymbol{\ell}_2$. Now consider how to compute a vector that is orthogonal to two vectors.

   (c) Let $\boldsymbol{\ell}_1$ be the line that connects $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$. Show that $\tilde{\boldsymbol{m}}_1 \times \tilde{\boldsymbol{m}}_2 = \boldsymbol{\ell}_1$.
   Same as part (b) but with the roles of points and lines reversed.

2. **Transformation Matrices**

   Compute a transformation matrix in homogeneous coordinates which describes a rotation around the y-axis through an angle of 45 degrees, followed by a translation with a vector $(-1, 2, -3)^T$, followed by a rotation around the x-axis through an angle of -60 degrees.
   Remember that order of transformations is important. Also note that for 3-D rotation around, say x-axis, the x coordinate of the original and transformed points will be same. Similarly for 3-D rotations around y or z axes.

3. **Stereo Reconstruction** Let us assume that we have two cameras in $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$. The extrinsic and intrinsic parameters of both cameras are given in the form of the matrices

   $$\boldsymbol{A}_1^{\mathbf{int}} = \boldsymbol{A}_2^{\mathbf{int}} = \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   $$\boldsymbol{A}_1^{\mathbf{ext}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_2^{\mathbf{ext}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -5 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Moreover, the focal length of both cameras is given by $f_1 = f_2 = 1$. Let us assume, that we have found the following correspondence:

$$\boldsymbol{x}_1 = \left(\frac{21}{2}, \frac{1}{2}\right)^T, \qquad \boldsymbol{x}_2 = \left(\frac{19}{2}, \frac{\sqrt{2}}{2}\right)^T.$$

Here, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ denote a point in pixel locations in the first and in the second frame, respectively. In order to restore the depth of the original scene point, perform the following steps:

(a) Compute the corresponding optical rays in the 3-D coordinate system of the cameras.
This only involves going from image coordinates to camera coordinates and should therefore involve the inversion of the intrinsic camera matrix. First represent the image coordinates $\boldsymbol{x}_i$ in homogenous form $\tilde{\boldsymbol{x}}_i$ and apply the inverse of the intrinsic matrix. This still yields points in $\mathbb{P}^2$ but represented in the camera coordinates system instead of the image coordinate system.

Since $f = 1$, the location of these points in $\mathbb{P}^3$ can be obtained by adding a depth component of $f = 1$. Finally, the optical rays in the camera coordinate system can be represented by multiplying these points by scalars λ_i for the 2 cameras. Changing λ_i will yield all points along the optical ray in camera i.

(b) Compute the corresponding optical rays in the 3-D Euclidean world coordinate system.
Camera to world involves applying the inverse of the extrinsic matrix. The rays from part (a) will now be represented in world coordinates.

(c) Intersect these rays to recover the 3-D world coordinates of the 3-D point that was projected on both image planes.
Find the value of λ_1 (and/or) λ_2. This will give you the depth along the corresponding optical ray from part (b). Therefore, we obtain the 3-D location of the point observed in the two images.