CS 565 Computer Vision

Nazar Khan

Lectures 6 and 7: Fourier Transform

Disclaimer

- Any unreferenced image is taken from the following web-page
  - http://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/

Note

- If a hammer is the only tool you have, you will look at every problem as a nail.
- The more tools you have, the more problems you can tackle.
- Our foray into the “Fourier world” is an attempt to gather as many tools as we can.

Fourier Transform

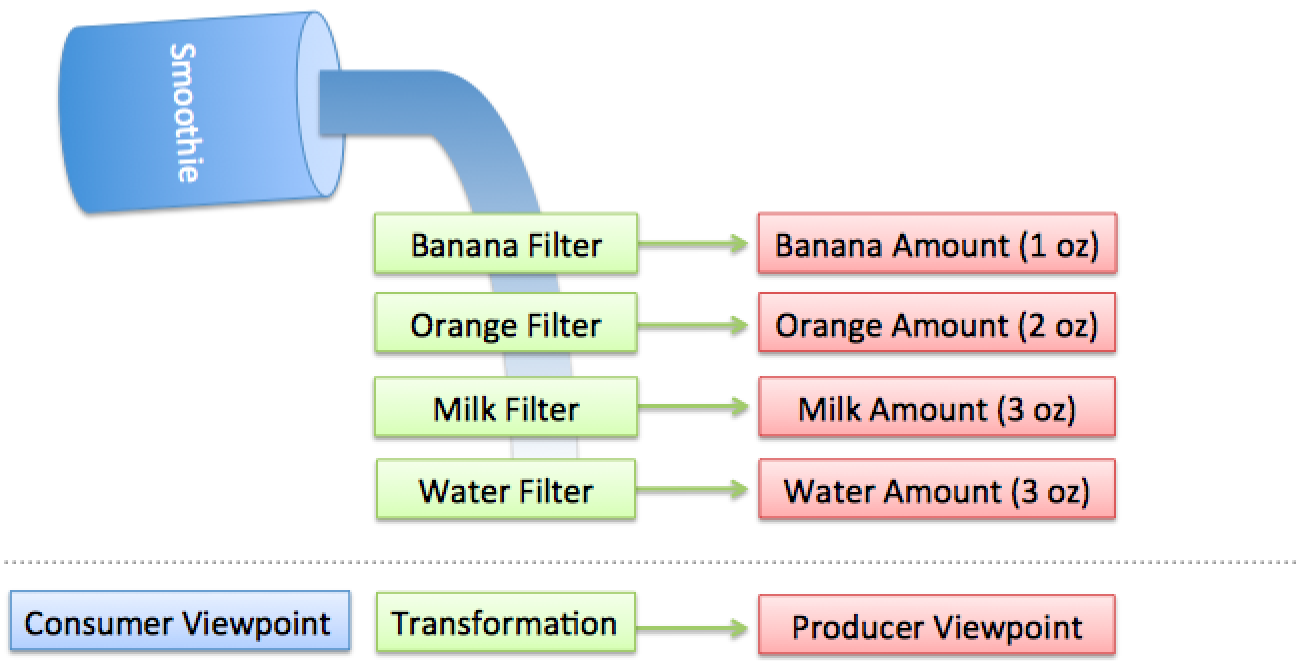
- One of the deepest mathematical insights.
- For any signal, it extracts its “ingredients”.
  - This is a **very powerful** idea.
  - Given an observation, it gives you the causes.
  - Given an image, it gives you its constituents.
- Understanding the Fourier Transform requires some of the **most beautiful mathematics** ever invented.

Fourier Transform

- The mathematics can become (more than) a little bit overwhelming.
- So we'll break it down into smaller, easier steps.

Fourier Transform – An Analogy

Smoothie to Recipe



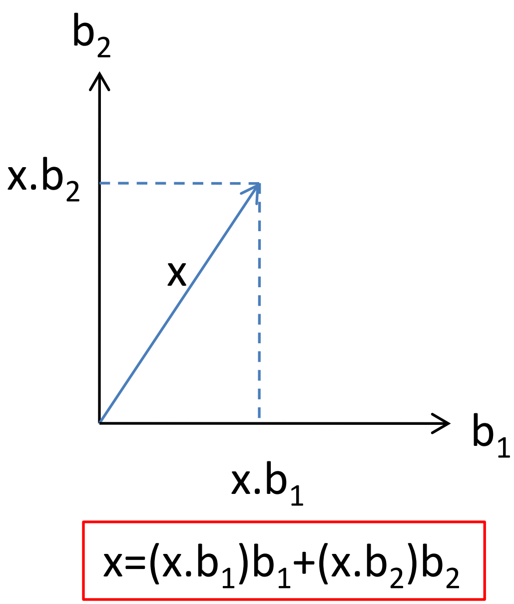
Source: http://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/

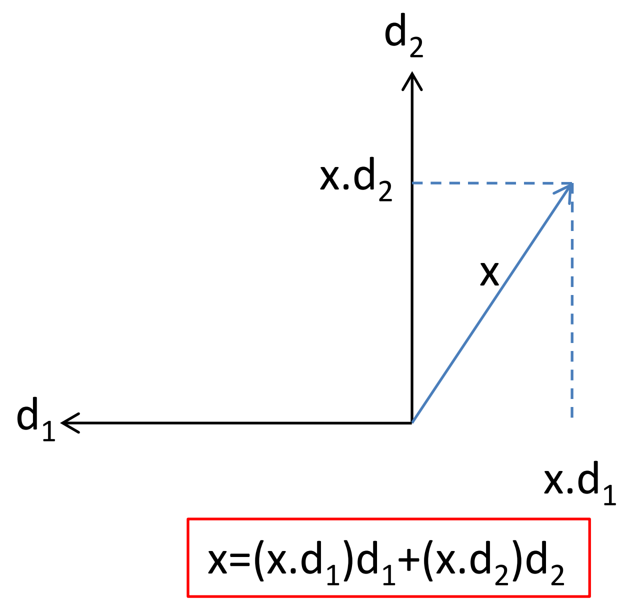
Fourier Transform

- We start with some pre-requisite mathematics.
  - Remember, math is not magic!
  - You **can** understand it if you take the correct perspective.

Projection

- A 2D vector x can be represented in an orthonormal basis $\{b_1,b_2\}$ by the formula $x=(x.b_1)b_1+(x.b_2)b_2$.
  - Coefficient for basis vector k is the <u>projection</u> $(x.b_k)$.





Mathematical Background

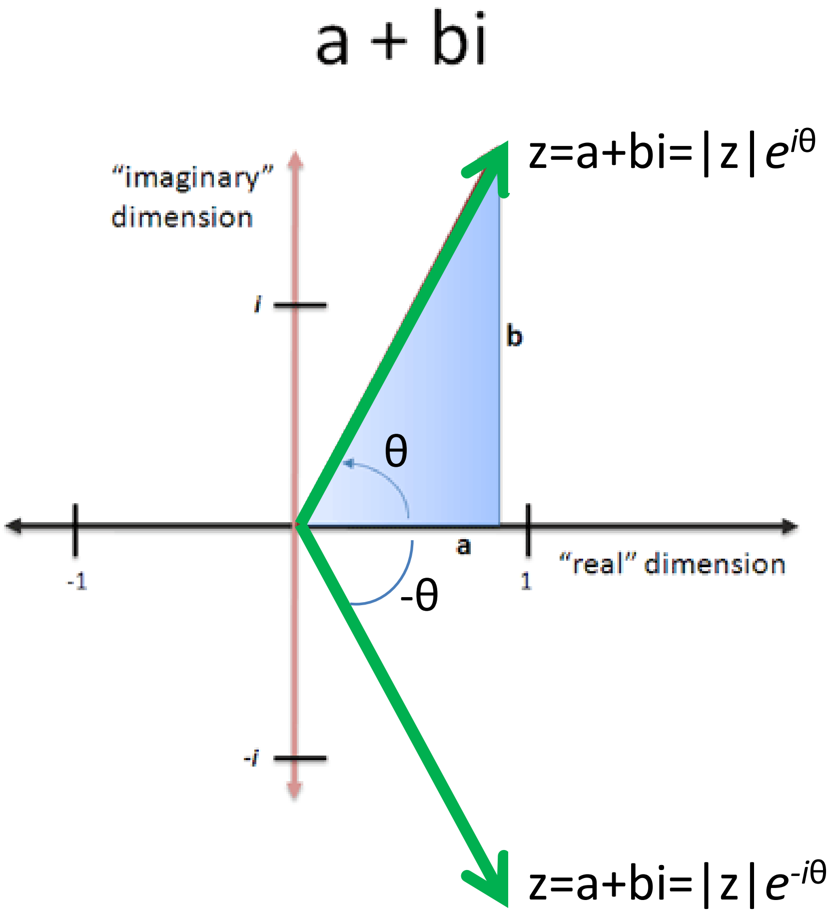
- *π*
  - circumference/diameter of **any** circle.
  - universal constant (π = 3.14159265...)
- *e*
  - Euler's number (e = 2.71828182...)
- *i*
  - non-existent, imaginary number (what!!!!)
  - makes analysis and computations easier ($i^2=-1$)

Complex Numbers

- Real numbers are represented by $\mathbb{R}^1$.
- We can write any real number x as x+0*i.*
- Therefore, $\mathbb{R}^1$ is contained within the space of complex numbers $\mathbb{C}^1$.
  - Complex numbers z have a real part Re(z) and an imaginary part Im(z).
- Basis vector for $\mathbb{R}^1$ is the scalar 1.
- Basis vectors for $\mathbb{C}^1$ are {(1,0),(0,*i*)}.

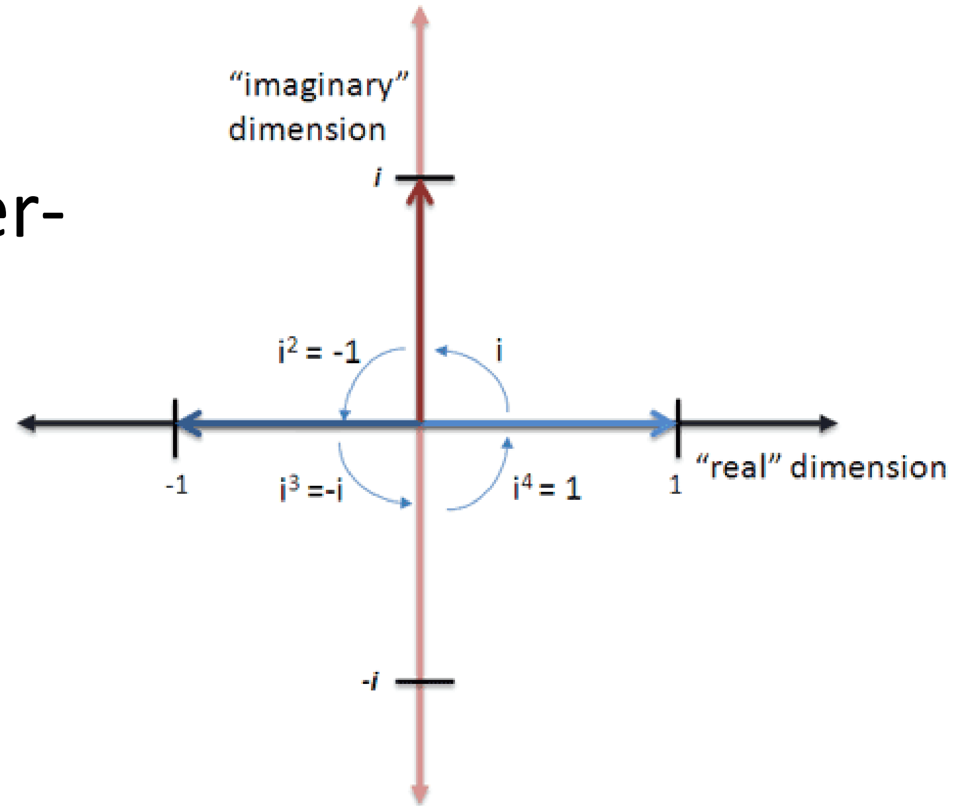
Complex Numbers

- Norm (magnitude, modulus) of z is given by $|z|=\mathrm{sqrt}(a^2+b^2)$.
- Phase is the angle $\theta=\arctan(b/a)$.
- A complex number can also be represented in Polar form $z=a+bi=|z|e^{i\theta}$.
- Conjugate of z is given by $\mathrm{conj}(z)=a-bi=|z|e^{-i\theta}$.
- **HW: Compute the values of sqrt(z*z) and sqrt(z*conj(z)). Which one yields the norm of z?**



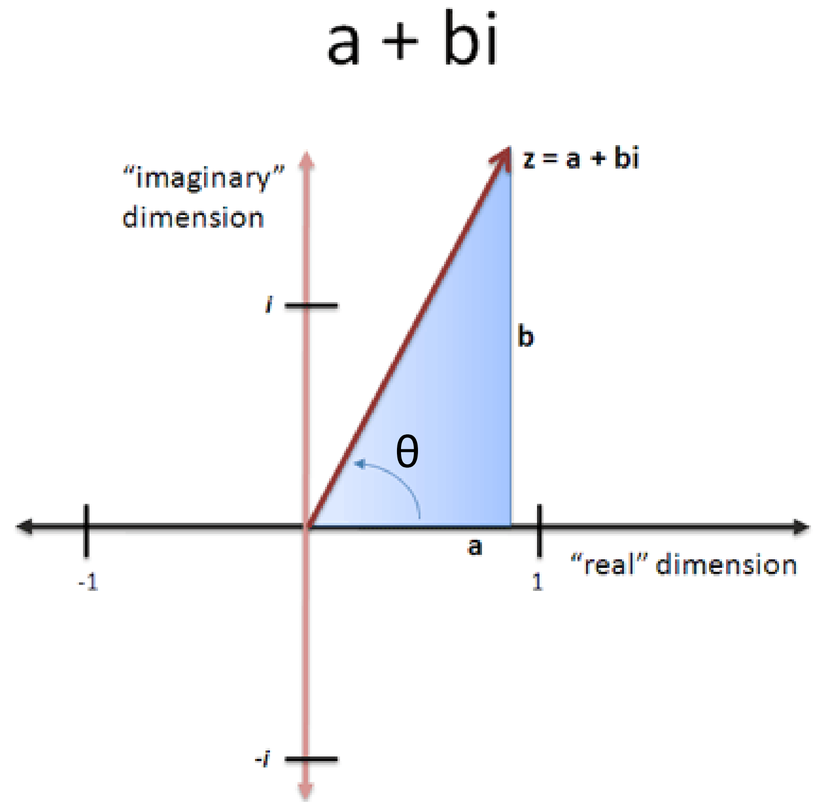
Multiplication by *i* Represents 90° Rotation in ℂ

- Multiplication by *i* is a **rotation** by 90° counter-clockwise in ℂ.
  - $1*i=i$
  - $1*i*i=-1$
  - $1*i*i*i=-i$
  - $1*i*i*i*i=1$



Multiplication by Complex Number Represents Rotation in ℂ

- Multiplication by any complex number $z = a+bi$ causes **rotation** by its **angle** $\theta = \arctan(b/a)$



The Bigger Picture

- The complex space $\mathbb{C}$ is just a generalization of the real space $\mathbb{R}$ where rotation amounts to multiplication.
- We don't care about $\mathbb{C}$ itself but we care about the fact that in $\mathbb{C}$ complicated rotations can be represented as simply as multiplications.
  - We don't care whether –ve numbers actually exist or not, we care that they make calculations of profit/loss or debit/credit easier.

Euler

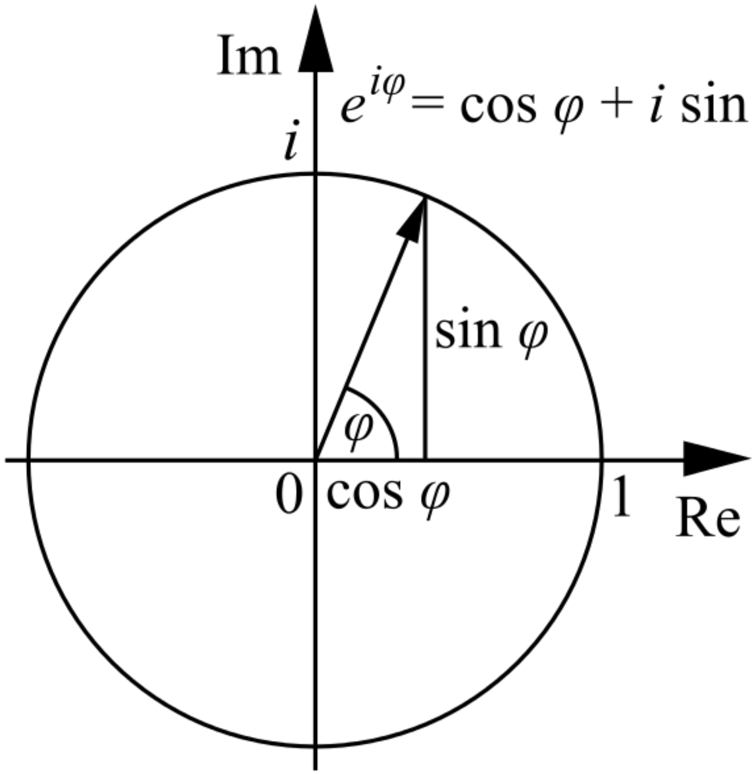
- One of the greatest mathematicians ever.
- Fundamental contributions in calculus, graph theory, optics, fluid dynamics, mechanics, astronomy and even music theory.
- Almost totally blind for the last 20 years of his life.
  - Yet did the most productive work during this time.

Source: http://en.wikipedia.org/wiki/Leonhard_Euler

Euler's Formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

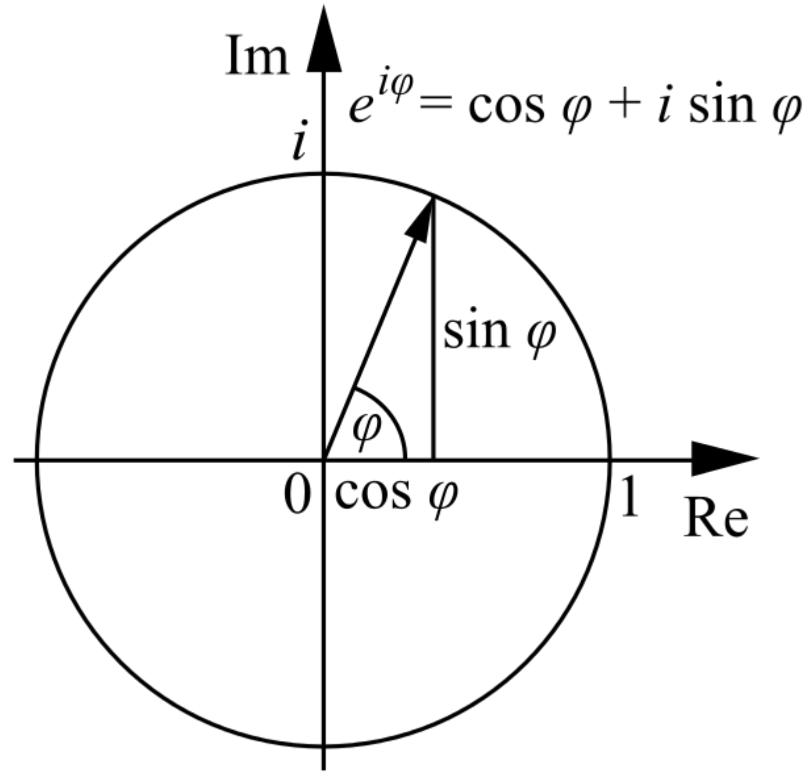
- **Mathematics does not get more beautiful than this equation.**
- What you can describe using sinusoids, you can describe using the numbers *e*=2.71828182… and *i*=sqrt(-1)



In 1988, readers of the *Mathematical Intelligencer* voted it "the Most Beautiful Mathematical Formula Ever“. In total, Euler was responsible for three of the top five formulae in that poll.

Euler's Formula

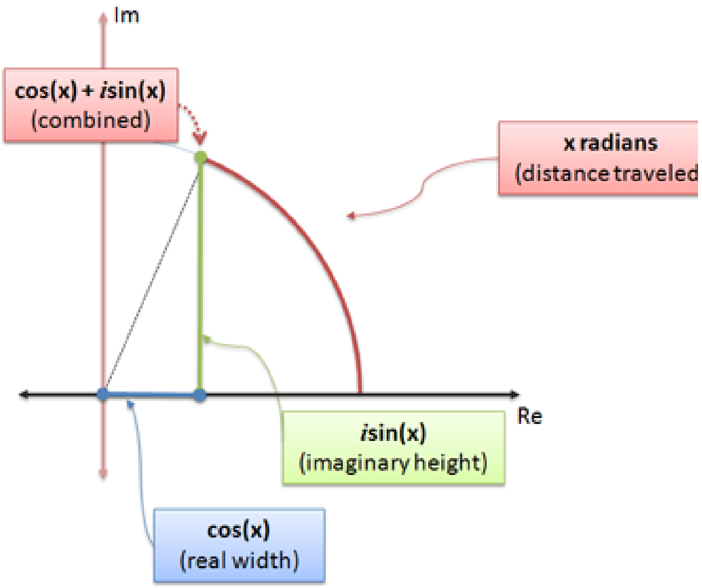
- What can we describe using *cos*(θ) and *sin*(θ)?
  - Positions on a circle.
- The formula says that that position is $2.7182818284^{\theta\sqrt{-1}}$ or simply $e^{i\theta}$.



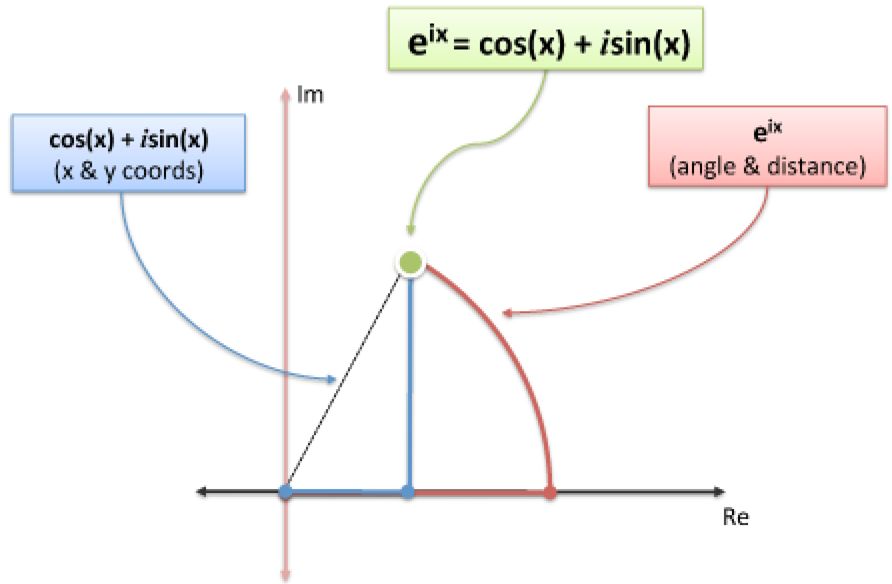

```
In Matlab:
>> [exp(sqrt(-1)*pi/4); cos(pi/4)+i*sin(pi/4)]
ans =
  0.7071 + 0.7071i
  0.7071 + 0.7071i
```

Euler's Formula

Traversing A Circle



Two Paths, Same Result



Euler’s Formula – The Bigger Picture

- **Describes circular motion.**
- Two ways to describe motion
  - Cartesian: Go 3 units east and 4 units north
  - Polar: Go 5 units at an angle of 71.56 degrees
- Depending on the problem, polar or Cartesian coordinates are more useful.
- **Euler’s formula lets us convert between polar and Cartesian representation to use the best tool for the job.**

The link between Euler's Formula and the Fourier Transform

- **Fourier's claim: Any signal can be made from circular motion.**
- Euler's formula generates <u>all</u> circular motions.
- So Euler's formula is the tool that the Fourier Transform needs to decompose signals into circular motions.

Fourier Transform

- In the Fourier Transform, we factorise the angular distance θ into angular speed *ω* and time *t*.
  - θ is angular distance along the circle (0—2*π*).
- Since $\theta = \omega t$, we can write $e^{i\theta} = e^{i\omega t}$
  - So $e^{i\omega t}$ determines how far we have moved along the circle in time *t* travelling at speed *ω.*
- By varying *ω* and *t*, we can compute how far a circular motion with speed *ω* will be at time *t*.

Fourier Transform

- Angular speed $\omega = 2\pi f$ where f is the frequency in cycles per unit time. (**HW: Verify this. Hint: Just look at the definitions and/or units of ω and f.**)
- So we can write $e^{i\theta} = e^{i\omega t} = e^{i2\pi ft}$
  - So $e^{i2\pi ft}$ determines how far we have moved along the circle in time t travelling with a frequency f.
- **By varying f and t, we can compute how far a circular motion with frequency f will be at time t.**

Fourier Transform

- Let $\mathbf{x} \in \mathbb{R}^N$. That is $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})^T$. Obviously, $\mathbf{x} \in \mathbb{C}^N$ too.
- Assume signal repeats after every 1 second.
- Divide this 1 second into N slices.
- Assume value x_n occurs at time t=n/N seconds.
- Position on the circle at time t=n/N is given by $e^{i\theta} = e^{i\omega t} = e^{i2\pi f t} = e^{i2\pi f n/N}$
- This gives us N positions along a circular motion with frequency f.
  - Let us denote them by the vector $\mathbf{u}_f = (e^{i2\pi f 0/N}, e^{i2\pi f 1/N}, \ldots, e^{i2\pi f (N-1)/N})^T$.
- Similarity of $\mathbf{x}$ and $\mathbf{u}_f$ can be computed by …
  - Inner-product $\mathbf{x}^T\mathbf{u}_f$.

Fourier Transform – The Bigger Picture

- Project signal $\mathbf{x}$ onto circular motions $\mathbf{u}_f$ of different frequencies $f = 0, 1, \ldots, N-1$.
  - Fourier coefficient for frequency f is $X_f = \mathbf{x}^T\mathbf{u}_f$

Fourier Transform – Projection onto Circular Motion

- For the Fourier transform, the N dimensional signal vector **x** is **projected** onto the circular basis vectors $\boldsymbol{e^{i2\pi f}}$.
  - Coefficient for basis vector with frequency *f* is the projection ($\mathbf{x}^{T}\boldsymbol{e^{-i2\pi f}}$).

$$X_f = \mathbf{x}^T e^{-i2\pi f}$$

$$= x_0 e^{-i2\pi f\,0/N} + \cdots + x_{N-1} e^{-i2\pi f \frac{N-1}{N}}$$

  - Do you notice something strange in the projection?

Fourier Transform – Projection onto Circular Motion

- Why the negative sign in the exponent?
- In order to measure lengths in any number space, a norm must be defined such that |x| = sqrt(x.x) = length of vector x.
- In the space of Complex numbers, inner product is defined as x.y = x*conj(y) where conj(y) = Re(y)-Im(y)*i* = |y|$\boldsymbol{e}^{-i\theta}$.
- **HW: For a complex vector f=(f_1,…,f_N), compute f.f and f.conj(f). Which one yields the squared norm of f (given by $|f|^2=|f_1|^2+\ldots+|f_N|^2$)?**
- The negative sign signifies conjugation of $e^{i2\pi f}$. So that the norm can be properly defined in Complex space.

Fourier Transform

$$X_f = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n e^{-i2\pi f\, n/N} = \frac{1}{\sqrt{N}} \mathbf{x}^T e^{-i2\pi f}$$

<u>Decompose</u> the signal <u>into</u> its constituent frequencies from *f*=0 to N-1.

Inverse Fourier Transform

$$x_n = \frac{1}{\sqrt{N}} \sum_{f=0}^{N-1} X_f e^{i 2\pi f \, n/N}$$

<u>Synthesize</u> the signal <u>from</u> its constituent frequencies.

Orthonormality of the Fourier Basis

- The basis vectors for different frequencies *f* are orthonormal.

Orthogonality Condition

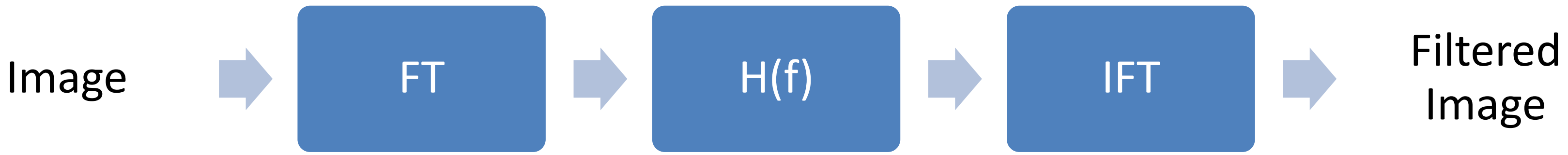
$$\begin{bmatrix} e^{i2\pi f_p \frac{0}{N}} \\ \vdots \\ e^{i2\pi f_p \frac{N-1}{N}} \end{bmatrix} \bullet \begin{bmatrix} e^{i2\pi f_q \frac{0}{N}} \\ \vdots \\ e^{i2\pi f_q \frac{N-1}{N}} \end{bmatrix} = 0 \quad \forall p \neq q$$

Normality Condition

$$\begin{bmatrix} e^{i2\pi f_p \frac{0}{N}} \\ \vdots \\ e^{i2\pi f_p \frac{N-1}{N}} \end{bmatrix} \bullet \begin{bmatrix} e^{i2\pi f_p \frac{0}{N}} \\ \vdots \\ e^{i2\pi f_p \frac{N-1}{N}} \end{bmatrix} = 1 \quad \forall p$$

- So the different frequencies do not interfere with each other in representing the signal.
- **HW: Prove orthonormality of Fourier basis.**

Frequency Domain Filtering Pipeline



Frequency Domain Low-Pass Filtering (Smoothing)

a b c

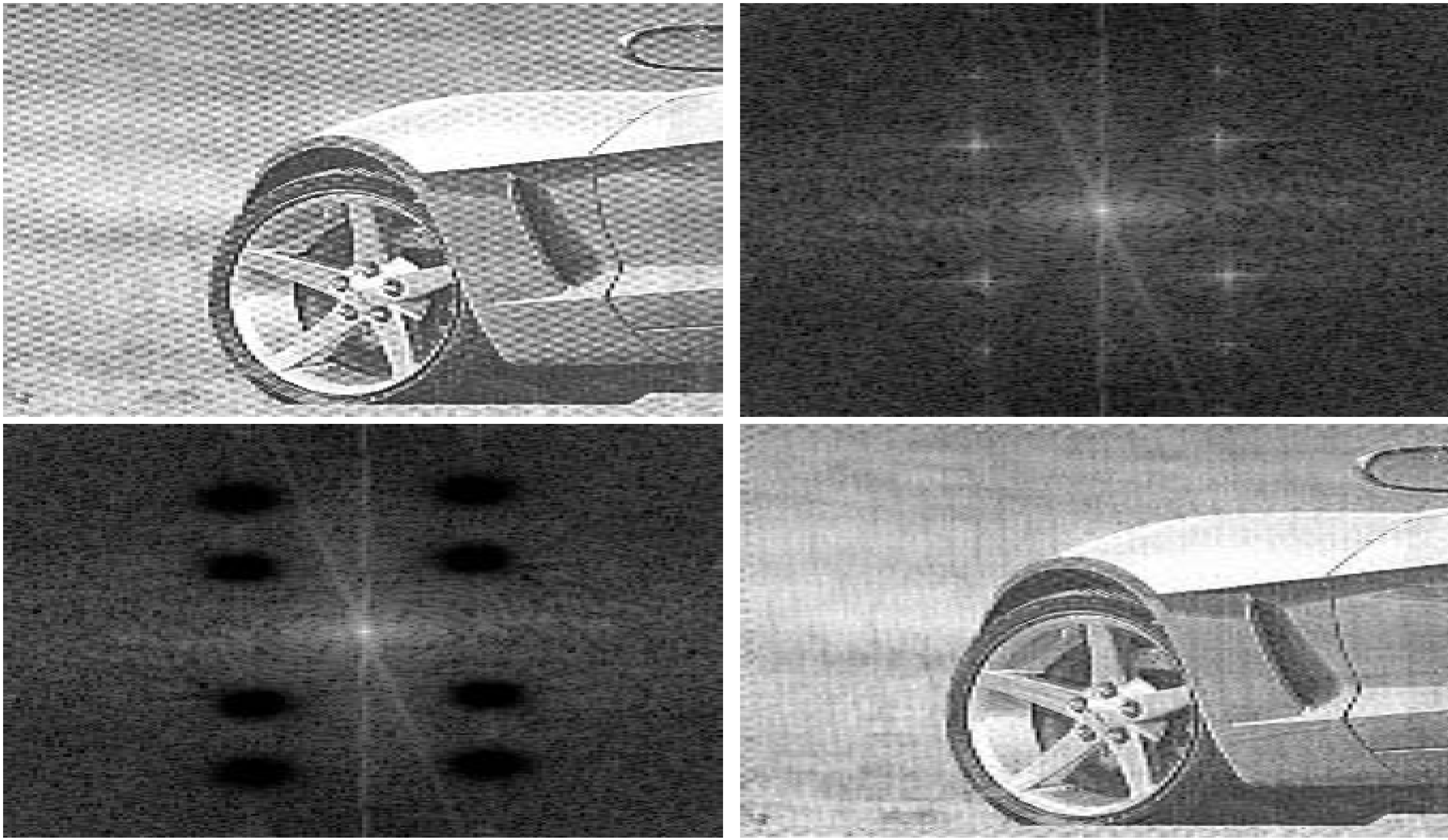
FIGURE 4.50 (a) Original image (784 × 732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

Source: Gonzalez & Woods

Frequency Domain High-Pass Filtering (Sharpening)

Source: Gonzalez & Woods

Frequency Domain Band-Pass Filtering



Source: Gonzalez & Woods