3 H in 4 tosses of a fair coin. — S was simple. P via counting.

→ 3H in 4 tosses of a biased coin? — HHHH vs TTTT ⇐ S is not simple.

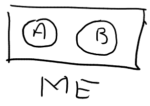
① Independence $\boxed{P(A \cap B) = P(A)P(B)}$

② Mutually Exclusive events (ME)

— If A has occured, then B definitely did not and cannot occur.

— H and T on coin toss are ME.

— 1, 2, 3, 4, 5, 6 on die roll are ME.



If A & B are ME, then $A \cap B = \emptyset$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

For ME events,

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

Are mutually exclusive events independent? Absolutely not!! They are dependent.

Bernoulli Experiment

Coin toss.

Output is either 0 or 1
T or H
fail or pass

Only 2 outcomes.

Bernoulli Process

A sequence of Bernoulli experiments.
coin tosses using the same coin.

$P(H)$ is same for each toss.

Binomial Experiment

A Bernoulli process of n coin tosses and recording the number of H's.

① Toss a coin n times.

② Record/count the number of heads.

$$S = \{0, 1, 2, \ldots, n\}$$

→ 3H in 4 tosses of a biased coin. ⟹ Binomial exp. with $n = 4$.

$P(H) = p$.

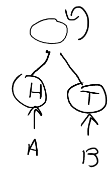
$P(H) = p$

Tossing a coin 4 times

$S_1 = \{$ HHHH, HHHT, HHTH, HTHH, THHH,
HHTT, HTTH, TTHH, HTHT, THHT,
THTH, HTTT, THTT, TTHT, TTTH,
TTTT $\}$ ← Sample space

$|S_1| = 2 \times 2 \times 2 \times 2$
$2^4 = 16$

Exp: way to toss ① coin



$P(\{3H \text{ in } 4 \text{ tosses}\}) = P(\{\underbrace{HHHT}_{A}, \underbrace{HHTH}_{B}, \underbrace{HTHH}_{C}, \underbrace{THHH}_{D}\})$ ← event space

$= P(A \text{ OR } B \text{ OR } C \text{ OR } D)$

ME

$= P(A \cup B \cup C \cup D)$

$= P(A) + P(B) + P(C) + P(D)$ ← bcz. A, B, C, D were M.E.

09999

$P(A) = P(\{HHHT\}) = P($ Head on 1st toss AND H on 2nd toss AND H on 3rd AND T on 4th $)$

$p(H \cup T) = p(S)$
$p(H) + p(T) = p(S) = 1$
$p + p(T) = 1$
$p(T) = 1 - p$

$= P($ H on 1st $\cap$ H on 2nd toss $\cap$ H on 3rd $\cap$ T on 4th $)$

$\boxed{\text{Independence}}$

$= P(\text{H on 1st})\, P(\text{H on 2nd})\, P(\text{H on 3rd})\, P(\text{T on 4th})$ ← bcz. coin tosses are independent.

$= p \times p \times p \times (1-p)$

$= p^3(1-p)$

$P(B) = P(\{HHTH\}) = P($ Head on 1st toss AND H on 2nd toss AND T on 3rd AND H on 4th $)$

$= P($ H on 1st $\cap$ H on 2nd toss $\cap$ T on 3rd $\cap$ H on 4th $)$

$\boxed{\text{Independence}}$

$= P(\text{H on 1st})\, P(\text{H on 2nd})\, P(\text{T on 3rd})\, P(\text{H on 4th})$ ← bcz. coin tosses are independent.

$= p \times p \times (1-p) \times p$

$= p^3(1-p)$

$P(C) = P(\{HTHH\}) = p^3(1-p)$

$P(D) = P(\{THHH\}) = p^3(1-p)$

So,

$$P(\{3H \text{ in } 4 \text{ tosses}\}) = \underset{HHHT}{P(A)} + \underset{HHTH}{P(B)} + \underset{HTHH}{P(C)} + \underset{THHH}{P(D)}$$

$$= p^3(1-p) + p^3(1-p) + p^3(1-p) + p^3(1-p)$$

$$= 4\, p^3 (1-p) = \binom{4}{3} p^3 (1-p)$$

By considering in how many ways can we get 3 H's (k) in 4 tosses (n).

_ _ _ _ $\binom{4}{3}$

$$P(\{2H \text{ in } 4 \text{ tosses}\}) = \binom{4}{2} p^2 (1-p)^2$$

$$= 6p^2(1-p)^2$$

C^4_2

$$P(k \text{ Heads in } n \text{ tosses of a coin with } p(H)=p) = \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

for $k = 0, 1, 2, \ldots, n$

Binomial Probability

Practice

Consider $n=4$ and $p=\frac{1}{2}$, $p=\frac{3}{4}$

#H	0	1	2	3	4
p(#H)					

(seq. of coin tosses)

Consider a $\underbrace{\text{Bernoulli process}}$.

① What is the prob. that k tails will be observed to get the 1st head?

$$p(k \text{ tails before 1st head}) = ? \qquad \text{for } k = ?$$

② What is the prob. that k tails will be observed to get the 10th head?

$$p(k \text{ tails before 10th head}) = ? \qquad \text{for } k = ?$$

③ What is the prob. that k trials will be observed to get the 1st head?

④ What is the prob. that k trials will be observed to get the 10th head?

- When does the experiment end?
- What would you have gotten on the last toss/trial?