Random Variables

Consider tossing a coin 3 times.

① Random Variable — A real-valued mapping or function.

$X(\underbrace{HHH}_{\text{event}}) = 3$

$S =$	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
X	3	2	2	2	1	1	1	0
Y	300	200	200	200	100	100	100	0
Z	300	150	150	150	0	0	0	−150

$\{X=0\} = \{TTT\}$ $\binom{3}{0}(1-p)^3$

$\{X=1\} = \{HTT, THT, TTH\}$ $\binom{3}{1}p(1-p)^2$

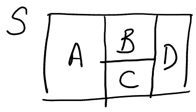
$\{X=2\} = \{HHT, HTH, THH\}$ $\binom{3}{2}p^2(1-p)$

$\{X=3\} = \{HHH\}$ $\binom{3}{3}p^3$

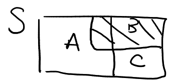
② A random variable partitions S.

Partition of S

- Non-overlapping subsets of S
- Union of those subsets gives S



← Partition of S



← Not a partition of S

Values of X	0	1	2	3
Probabilities	$(1-p)^3$	$3p(1-p)^2$	$3p^2(1-p)$	p^3

③ Probability Density Function of X

= Probability Mass Function of X

= Probability Distribution of X

= Density of X

- Density can be computed using the rules of probability. You do not need to perform the experiment.

Types of RVs

Discrete

Partitions S into countably many disjoint events.

(countably many: Finite / Countably infinite)

Example

Coin toss $\{H, T\}$

Die roll $\{1, 2, 3, 4, 5, 6\}$

X	a_1	a_2	$\cdots$	a_n	$\cdots$
$P(X)$	P_1	P_2	$\cdots$	P_n	$\cdots$

(i) $P_i \geq 0 \quad \forall i$

(ii) $P_1 + P_2 + \cdots = 1$

Note that (i) & (ii) imply that $P_i \leq 1 \ \forall i$.

For any subset $A \in S$,

$$P(X \in A) = \sum_{a \in A} P(X = a)$$

$P(X = a)$ — X: Random Variable; a: Value of the random variable.

For discrete R.V
Density = Probability

For continuous R.V
Density ≠ Probability

Continuous

Partitions S into uncountably infinite sets.

Example

Randomly selecting a number from $[0,1]$.

(i) $f(x) \geq 0 \quad \forall x$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

$f(x)$ is called the density. ≠ probability.

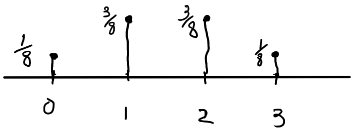
④

Values of X	0	1	2	3
Probabilities	$(1-p)^3$	$3p(1-p)^2$	$3p^2(1-p)$	p^3
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Assume $p = \frac{1}{2}$

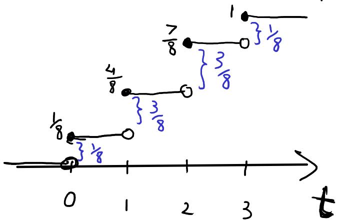
Cumulative Distribution Function

$$\boxed{F(t) = P(X \le t)\ ,\ t \in \mathbb{R}}$$



PDF

Adding $\longrightarrow$



CDF

Probability Density Function $\longrightarrow$ Add $\longrightarrow$ Cumulative Distribution Function

$P(X<0) = P(\{\emptyset\}) = 0$

$F(0) = P(X \le 0) = \underbrace{P(X<0)}_{0} + \underbrace{P(X=0)}_{\frac{1}{8}}$

$= \frac{1}{8}$

$P(X<1) = \underbrace{P(X<0)}_{0} + \underbrace{P(0 \le X < 1)}_{P(\{X=0\})}$

$= 0 + \frac{1}{8} = \frac{1}{8}$

$P(X \le 1) = \underbrace{P(X<1)}_{\frac{1}{8}} + \underbrace{P(X=1)}_{\frac{3}{8}}$

$= \frac{4}{8}$

$$\text{Density} \underset{\text{Subtract}}{\overset{\text{Add}}{\rightleftarrows}} \text{CDF}$$

$F(0) - F(0^-) = \frac{1}{8} - 0 = \frac{1}{8} = P(X=0)$

$F(1) - F(1^-) = \frac{4}{8} - \frac{1}{8} = \frac{3}{8} = P(X=1)$

$F(2) - F(2^-) = \frac{7}{8} - \frac{4}{8} = \frac{3}{8} = P(X=2)$

$F(3) - F(3^-) = 1 - \frac{7}{8} = \frac{1}{8} = P(X=3)$

CDF $\longrightarrow$ Subtracting $\longrightarrow$ PDF

CDF properties

(i) CDF $F(t)$ is a non-decreasing function of t.

(ii) $F(-\infty) = 0$ and $F(\infty) = 1$

$P(X \le -\infty) = 0$ $\qquad$ $P(X \le \infty) = 1$

$-\infty \le X \le \infty$

$-\infty \le \{0, 1, 2, 3\} \le \infty$

(iii) $F(t)$ is right-continuous

(iv) $P(X = a) = F(a) - F(a^-)$