Discrete Random Variables
↓
Integer

$X \rightarrow$ real-valued function

domain is S — range is a countable set Δ (del)

For each $a \in \Delta$, the resulting partitioning subset of S is $\{X = a\}$

$$f(a) = P(X=a) \quad \text{for } a \in \Delta$$

↑
density of random variable X.

$$\boxed{\underset{\text{domain}}{S} \xrightarrow{X} \underset{\text{range} \quad \text{domain}}{\Delta} \xrightarrow{f} \underset{\text{range}}{[0,1]}}$$

Coin toss 2 times

$S = \{HH, HT, TH, TT\}$

$X = \#$ of heads

$\Delta = \{0, 1, 2\}$

X	0	1	2
f	$(1-p)^2$	$2p(1-p)$	p^2
	$f(0)$	$f(1)$	$f(2)$

Density is represented in different ways.

$f(a)$, $f_X(a)$, $p(a)$, $p_X(a)$

Binomial Random Variable

Coin toss 2 times

$S = \{HH, HT, TH, TT\}$

$X = \#$ of heads

$\Delta = \{0, 1, 2\}$

X	0	1	2
f	$(1-p)^2$	$2p(1-p)$	p^2
	$f(0)$	$f(1)$	$f(2)$

$$f(k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k \in \Delta = \{0,1,2,\dots,n\}$$

$$X \sim Bin(n, p)$$

parameters

X	0	1	2	⋯	n
f	–	–	–	⋯	–

Avoid this

Cumulative Distribution Function

CDF

$$F(t) = P(X \le t) = \sum_{k=0}^{t} f(k) = \sum_{k=0}^{t} \binom{n}{k} p^k (1-p)^{n-k}$$

There is no closed-form expression for CDF of Binomial Random Variable.

$F(1) = f(0) + f(1)$
$\quad = (1-p)^2 + 2p(1-p)$

$F(2) = 1$

$F(0) = (1-p)^2$

$F(-1) = 0$

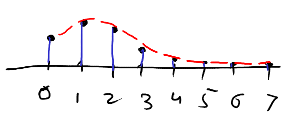
Roll-nos. 1–8

$X \sim Bin(7, 0.2)$

X	0	1	2	3	4	5	6	7
f	0.21	0.36	0.275	0.114	0.028	0.0043	0.0003	0.000013

$$f(4) = P(X=4) = \binom{7}{4} (0.2)^4 (0.8)^{7-4}$$
$$= 0.028$$

f

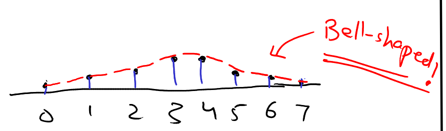


Roll nos. 9–16

$X \sim Bin(7, 0.5)$

X	0	1	2	3	4	5	6	7
f	0.0078	0.054	0.164	0.273	0.273	0.164	0.054	0.0078

f

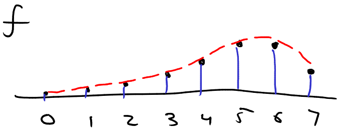


Roll nos. 17–24

$X \sim Bin(7, 0.75)$

X	0	1	2	3	4	5	6	7
f	6.1×10^{-5}	0.0012	0.0115	0.0576	0.172	0.3114	0.3115	0.1335

f



$$\frac{2^k}{k!} \qquad\qquad \frac{3^k}{k!}$$

Roll nos. 30–40

$k = 0 \to 10$

Roll nos. 40–50

$k = 0 \to 10$

k	0	1	2	3	4	5	6	7	8	9	10
$\frac{2^k}{k!}$	1	2	2	$\frac{8}{6}$	$\frac{2}{3}$	$\frac{4}{15}$	$\frac{4}{45}$	$\frac{8}{315}$	$\frac{2}{315}$	$\frac{4}{2835}$	$\frac{4}{14175}$

$$\sum_{k=0}^{10} \frac{2^k}{k!} = 7.38 \qquad\qquad \sum_{k=0}^{10} \frac{3^k}{k!} = 20.04$$

$e = 2.71828182846$

$$e^2 = 7.389056 \qquad\qquad e^3 = 20.085$$

$a \geq 0$, $b \geq 0$, $a+b$, $\frac{a}{a+b}$, $\frac{b}{a+b}$

$$\frac{a}{a+b} + \frac{b}{a+b}$$

between 0 & 1

$$\frac{a+b}{a+b} = 1$$

sum $= 1$

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$$

$$\boxed{P(X = k) = \frac{1}{e^{\lambda}} \frac{\lambda^k}{k!}, \quad k = \{0, 1, 2, 3, \ldots\}}$$

Poisson Random Variable

$X \sim \text{Poisson}(\lambda)$

parameter $\lambda > 0$

① Telephone Exchange

Model the Numbers of calls arriving in

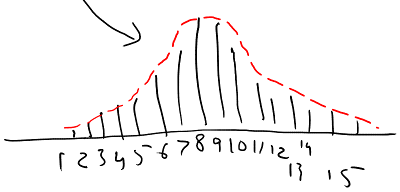
② Radiative material

Number of particles emitted by a radioactive substance.

k	0	1	2	3	4	5	6	7	8	9	10
$\frac{2^k}{k!}$	1	2	2	$\frac{8}{6}$	$\frac{2}{3}$	$\frac{4}{15}$	$\frac{4}{45}$	$\frac{8}{315}$	$\frac{2}{315}$	$\frac{4}{2835}$	$\frac{4}{14175}$
$f(k)$	$\frac{1}{e^2}$	$\frac{2}{e^2}$	...								

$X \sim \text{Poisson}(2)$

$X \sim \text{Poisson}(0.9)$



CDF has no closed-form expression.

$$F(t) = P(X \leq t) = e^{-\lambda} \sum_{k=0}^{t} \frac{\lambda^k}{k!}$$

Consider a Bernoulli process (seq. of coin tosses).

1. What is the prob. that k tails will be observed to get the 1st head?
   P(k tails before 1st head) = ? for k = ?
2. What is the prob. that k tails will be observed to get the 10th head?
   P(k tails before 10th head) = ? for k = ?
3. What is the prob. that k trials will be observed to get the 1st head?
4. What is the prob. that k trials will be observed to get the 10th head?

In a Bernoulli process,

1. X = number of tails before the 1st head

   $X \sim$ Geometric(p) $\quad \Delta = \{0, 1, 2, 3, \dots\}$

   $P(X=k) = (1-p)^k p$

2. Y = number of tails before ~~the~~ 10th head

   $Y \sim$ Negative Binomial(p) $\quad \Delta = \{0, 1, 2, \dots\}$

   $P(Y=k) = \binom{k+9}{9}(1-p)^k p^{10}$ $\quad \underbrace{- \; - \; \dots \; -}_{k+9} \; \underline{H}$

3. Z = number of trials before the 1st head

   $Z \sim$ Shifted Geometric(p) $\quad \Delta = \{1, 2, 3, \dots\}$

   $P(Z=k) = (1-p)^{k-1} p$

4. W = number of trials to get the 10th head

   $W \sim$ Shifted Negative Binomial(p) $\quad \Delta = \{10, 11, 12, \dots\}$

   $P(W=k) = \binom{k-1}{9}(1-p)^{k-10} p^{10}$

k tails, 10 heads

$\underbrace{- \; - \; - \; - \; -}_{k+9} \; \underline{H}$

$\binom{k+9}{9}$

$\underbrace{- \; - \; -}_{k-1} \; \underline{H}$ (9 H)

$\binom{k-1}{9}$

Geometric Series

$a + \underbrace{ar}_{r} + \underbrace{ar^2}_{r} + \underbrace{ar^3}_{r}$

X	0	1	2	3	...
P(X)	p	$(1-p)p$	$(1-p)^2 p$	$(1-p)^3 p$	
	a	ar	ar^2	ar^3	