Expectation

Often an entire distribution of a R.V is summarized using 2 numbers

① Centre

② Spread

Both numbers are computed using an operation called "expectation".

$$E(X) = \sum_{x\in\Delta} x \overbrace{P(X=x)}^{f(X=x)}$$ ← for discrete X

$$E(X) = \int_{x\in\Delta} x f(x) dx$$ ← for continuous X.

Example

One roll of a fair die.

X	1	2	3	4	5	6
P(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$

$$= 3.5$$

E(X) can be outside Δ.

One roll of a biased die.

X	1	2	3	4	5	6
P(X)	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$

$$E(X) = 1\times\frac{1}{12} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{4}$$

$$= 3.9$$

Expectation E(X) is sometimes also called the average or mean.

Average(1, 2, 3, 4, 5, 6)

$$= \frac{1+2+3+4+5+6}{6} = 3.5 \leftarrow$$ Ordinary average

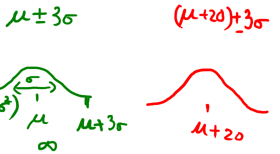
$$= 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + \dots + 6\times\frac{1}{6} = 3.5 \leftarrow$$ When X ~ U then E(X) = ordinary average.

But when $X \nsim U$ then E(X) is a weighted average.

Properties of Expectation

① E(X) is a real number usually denoted μ or μ_X.
(bcz Δ contains real numbers)

X ~ N(μ, σ²)



$$E(X) = \int_{-\infty}^{\infty} x \mathcal{N}(x;\mu,\sigma^2)\, dx$$

$$= \mu$$
↑ average
└ center

② E(X) will always lie between the range of Δ.

1 – 6

1	2	3	4	5	6
0.01	0.01	0.01	0.01	0.95	0.01

$$E(X) = 16(0.01) + 5(0.95)$$

$$= 4.75 \leftarrow$$ (close to 5)

1 – 6

$5 \times \cancel{2}$

1 – 6

③ $E(\overbrace{\underbrace{aX+b}_{\text{random}}}^{f(X)}) = \sum_{x\in\Delta} (ax+b) P(\underbrace{ax+b})$

random bcz. X is random

کب آئے گا؟

جب x آئے گا۔

ax+b کا chance کتنا ہے؟

جتنا x کا chance ہے۔

Y = aX + b

P(Y = ax + b) = P(X = x)

P(Y = a(3)+b) = P(X = 3)

$$= \sum_{x\in\Delta} (ax+b) P(X=x)$$

$$= \sum_{x\in\Delta} ax P(X=x) + \sum_{x\in\Delta} b P(X=x) = a \overbrace{\sum_{x\in\Delta} x P(X=x)}^{E(X)} + b \overbrace{\sum_{x\in\Delta} P(X=x)}^{1} = aE(X)+b$$

2x + 2y + 2z
2(x + y + z)

$$E(aX+b) = aE(X)+b \quad \longleftarrow \quad \text{Linearity Property}$$

X	1	2	3	4	5	6
$p(x)$	0.01	0.01	0.01	0.01	0.95	0.01
$2X-3$	-1	1	3	5	7	9

$a=2$
$b=-3 \rightarrow 2X-3$

$\boxed{2x-3} = ① \quad$ Chance ?

$2x = 3+1$

$\boxed{x} = \frac{4}{2} = ② \quad$ Chance ?

$X \sim \text{Bin}(n,p) \qquad \Delta = \{0,1,2,\ldots,n\}$

$$E(X) = \sum_{x\in\Delta} x \underbrace{\binom{n}{x} p^{x}(1-p)^{n-x}}_{P(X=x)}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^{x}(1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^{x}(1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^{x}(1-p)^{n-x}$$

$$\frac{x}{x!} = \frac{\cancel{x}}{\cancel{x}(x-1)!} = \frac{1}{(x-1)!}$$

Game: Sum of prob. is 1

$$\sum \frac{n!}{x!(n-x)!} p^{x}(1-p)^{n-x} = 1$$

$y = x-1$
when $x=1$
$y = 1-1 = 0$

Let $y = x-1 \Rightarrow x = y+1$
$m = n-1 \Rightarrow n = m+1$

$n-x$
$(m+1)-(y+1)$
$= m-y$

$$= \sum_{y=0}^{m} \frac{(m+1)!}{y!\,(m-y)!} p^{y+1}(1-p)^{m-y}$$

$$(m+1)! = (m+1)\,m!$$

$$= \sum_{y=0}^{m} \frac{(m+1)\,m!}{y!\,(m-y)!}\cdot p\cdot p^{y}(1-p)^{m-y}$$

$\binom{m}{y}$

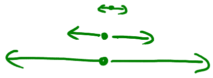
$$= (m+1)p \underbrace{\sum_{y=0}^{m} \binom{m}{y} p^{y}(1-p)^{m-y}}_{=1} = (m+1)p = np$$

$X \sim \text{Bin}(n,p)$

$$E(X) = np$$

$X \sim \text{Bin}(n,p)$
approximated by $\mathcal{N}(\underset{\mu}{\underset{\uparrow}{np}}, \underbrace{np(1-p)}_{?\ \sigma^2})$

Next lecture



σ — spread

Fair Die

$E(X) = 3.5$

X_1
X_2
X_3
$\vdots$

$$\frac{X_1+X_2+\cdots+X_{10}}{10} = 3.7$$

$$\frac{X_1+X_2+\cdots+X_{100}}{100} = 3.54$$

$$\frac{X_1+X_2+\cdots+X_{1000}}{1000} = 3.503$$

Biased Die

$E(X) = 3.9$

X_1
X_2
X_3
$\vdots$

$$\frac{X_1+X_2+\cdots+X_{10}}{10} = 3.7$$

$$\frac{X_1+X_2+\cdots+X_{100}}{100} = 3.86$$

$$\frac{X_1+X_2+\cdots+X_{1000}}{1000} = 3.912$$

Law of large numbers

As $N \to \infty$, ordinary average $\to E(X)$.