Simple sample spaces
- Counting methods
  - Multiplication Principle
  - Permutation
  - Combination

$$C^n_k = \frac{P^n_k}{k!}$$

a b c d

$\overline{abc}$: abc, acb, bac, bca, cab, cba — $3!$

$\overline{abd}$ → $3!$

$\overline{acd}$ → $3!$

$\overline{bcd}$ → $3!$

$\underbrace{3! \quad 3! \quad 3! \quad 3!}_{\text{24 permutations}}$

4 groups
4 combinations
$C^4_3 = \frac{4!}{1!\,3!} = 4$

Q. A box of apples.

G good apples.

B bad apples.

We reach in and draw n apples all at once.
What is the probability that we get k good apples?

Ans. $S = \{ \underbrace{\_\ \_\ \_ \cdots \_}_{n\text{-tuple}}, \underbrace{\_\ \_\ \_ \cdots \_}_{n\text{-tuple}}, \dots \}$

When does the experiment end?

When you get n apples.

Is S simple?
Yes

$$|S| = C^{B+G}_n = \binom{B+G}{n}$$

$\to E = \{$ k good apples and n−k bad apples $\}$

To find $|E|$, imagine if you were "forced" to form event E, then how many ways could we have?

$\cancel{\binom{n}{k}}$ $\quad |E| = \binom{G}{k} \times \binom{B}{n-k}$

$\underbrace{\underbrace{B\ B\ B \cdots B}_{n-k}\ \underbrace{G\ G\ G \cdots G}_{k}}_{n}$

$$P(E) = P(k \text{ good apples in } n \text{ apples}) = \frac{|E|}{|S|} = \frac{\binom{G}{k} \times \binom{B}{n-k}}{\binom{G+B}{n}}, \quad k = 0, 1, 2, \dots, \min(G, n)$$

Hypergeometric model

Q. Consider a well-shuffled deck of 52 playing cards.
I randomly draw 5 cards.
Find the probability of getting 2 pairs.

52: ♡ Heart 13, ♢ Diamond 13, ♤ Spade 13, ♧ Club 13

$\underbrace{A, 2, 3, \dots, 10, J, Q, K}_{13}$

♤ ♧ numbers?

$\underbrace{2\ \ 6\ \ 2}_{\text{pair}}\ \ J\ \ K \ \ \cancel{\in}\ E$

$2\ \ 6\ \ 2\ \ J\ \ 6 \leftarrow E$

pair, pairs

$2\ \ 2\ \ 2\ \ J\ \ K$

3 pairs

13: A, A, A, A; ~~2, 2, 2, 2~~ $\leftarrow \binom{4}{2}$; 3, 3, 3, 3; ⋮; ~~J, J, J, J~~ $\leftarrow \binom{4}{2}$; Q, Q, Q, Q; K, K, K, K

$\binom{13}{2}$

$S = \{ \underbrace{\_\ \_\ \_\ \_\ \_}_{5\text{-tuple}}, \underbrace{\_\ \_\ \_\ \_\ \_}_{5\text{-tuple}}, \dots \}$

Is S simple?
Yes

$$|S| = \binom{52}{5}$$

Imagine being "forced" to make set E.
How many ways?

4 cards picked

$$|E| = \binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times \binom{44}{1}$$

choose 2 categories from 13; choose 2 cards from 4 of 1st category; choose 2 cards from 4 of 2nd category; choose 1 from 44 cards.

Stage 1: pick the pair categories

Stage 2: pick 2 cards from each category

Stage 3: pick last card from from the remaining categories.

$$P(E) = \frac{|E|}{|S|} = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}}$$

Q. Roll a die 10 times and record their outcomes.

$$S = \{ \underbrace{----------}_{10\text{-tuple}}, \ldots \}$$

S is simple.

$$|S| = 6^{10} = \underline{6} \times \underline{6} \times \underline{6} \times \underline{6} \times \underline{6} \times \underline{6} \times \underline{6} \times \underline{6} \times \underline{6} \times \underline{6}$$

Event E is 6 repeated 2 times
1 repeated 3 times
2 ——— 5 ———

$P(E) = ?$

$$|E| = \binom{10}{2} \times \binom{8}{3} \times \binom{5}{5}$$

6 6 6 6 6 6

$$= \binom{10}{2} \times \binom{10-2}{3} \times \binom{10-2-3}{5} = \frac{10!}{8!\,2!} \times \frac{8!}{5!\,3!} \times \frac{5!}{0!\,5!}$$

$$= \frac{10!}{2!\,3!\,5!} \longleftarrow \frac{n!}{k_1!\,k_2!\,k_3!\ldots k_m!} = \binom{n}{k_1, k_2, \ldots, k_m}$$

multinomial Coefficient

$$P(E) = \frac{10!}{2!\,3!\,5!\,6^{10}}$$

$$C^n_k = \binom{n}{k}$$

Binomial Coefficient