Experiment 1

Consider a fair coin toss.

$S_1 = \{H, T\}$

$A = \{H\}$

$p(A) = \frac{1}{2}$

Experiment 2

Roll a fair die

$S_2 = \{1, 2, 3, 4, 5, 6\}$

$B = \{4\}$

$p(B) = \frac{1}{6}$

Consider the event {H on coin toss <u>and</u> 4 on die roll}

– Which sample space does this event belong to? $\underbrace{S_1?}_{\text{No}}$ $\underbrace{S_2?}_{\text{No}}$ Neither

It is the outcome of a <u>3rd experiment</u> with sample space

$$S_3 = \left\{ \begin{array}{l} (H,1), (H,2), (H,3), (H,4), (H,5), (H,6), \\ (T,1), (T,2), (T,3), (T,4), (T,5), (T,6) \end{array} \right\}$$

Joint probability

$\longrightarrow$ P({H on coin toss <u>and</u> 4 on die roll}) $= P(\{(H,4)\}) = P(H,4) = P(H \underset{\text{and}}{\cap} 4) = \frac{1}{12}$

<u>Notice</u> that $\underset{\frac{1}{12}}{P(H \cap 4)} = \underset{\frac{1}{2}}{P(H)} \times \underset{\frac{1}{6}}{P(4)}$

marginal Prob.   marginal Prob.

Q1. Which of the 3 experiments will take the most effort?

L Experiment

Q2. Does Exp. 1 <u>influence</u> Exp. 2 in <u>any</u> way? <u>No</u>

Exp. 2 ... Exp. 1 <u>No.</u>

influence: increase, decrease or block the chances

Such experiments that do not influence each other are called <u>independent</u>.

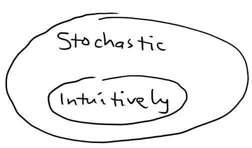
– Output of the coin toss is <u>independent</u> of the output of the die roll and vice verse.

For independent events A and B, their joint probability $P(\underbrace{A \cap B}_{S_3}) = P(A,B) = P(AB)$ can be computed via the smaller sample spaces S_1 and S_2 as

$$\underset{\text{Joint}}{P(A \cap B)} = \underset{\text{Marginal}}{P(A)} \; \underset{\text{Marginal}}{P(B)}$$

– Independent events are also called probabilistically / stochastically independent.

– Sometimes independence can be seen <u>intuitively</u> (such as coin & die example) but sometimes it's not <u>intuitively obvious</u>.



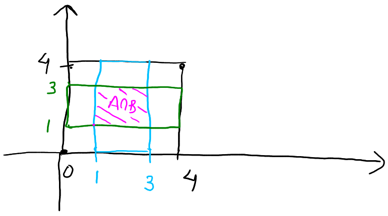
Q. Let a point be selected at random from a square $\underbrace{[0,4]}_{x\text{-range}} \times \underbrace{[0,4]}_{y\text{-range}}$.

Let A be the event that selected point lies inside the rectangle $[1,3] \times [0,4]$.

Let B be the event that selected point lies inside the rectangle $[0,4] \times [1,3]$.

Are events A and B independent?

S



For independence

$P(A \cap B) = P(A)\,P(B)$

Let's see if that holds. $\leftarrow$

$$P(A) = \frac{\text{size}(A)}{\text{size}(S)} = \frac{\text{area}(A)}{\text{area}(S)} = \frac{(3-1)\times(4-0)}{(4-0)\times(4-0)} = \frac{8}{16} = \frac{1}{2}$$

$$P(B) = \frac{1}{2}$$

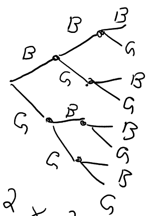
$$P(A\cap B) = \frac{\text{area}(A\cap B)}{\text{area}(S)} = \frac{(3-1)\times(3-1)}{16} = \frac{4}{16} = \frac{1}{4}$$

Since $P(A\cap B) = \frac{1}{4} = \frac{1}{2}\times\frac{1}{2} = P(A)\,P(B)$, A and B are <u>independent events</u>.

Q. Consider randomly selecting a family of 3 children.

$S = \{BBB, BBG, BGB, GBB, GGB, GBG, BGG, GGG\}$

$|S| = \underline{2}\times\underline{2}\times\underline{2} = 2^3 = 8$



$2\times 2\times 2 = 2^3$

Assume S is simple.

A = { family has both boys and girls }

B = { family has at most one girl }.

Are A & B independent events?

$A = \{BBG, BGB, GBB, GGB, GBG, BGG\} \Rightarrow |A| = 6$

$B = \{BBB, BBG, BGB, GBB\} \Rightarrow |B| = 4$

$P(A) = \frac{6}{8}$

$P(B) = \frac{4}{8}$

$A\cap B = \{BBG, BGB, GBB\} \Rightarrow |A\cap B| = 3$

$P(A\cap B) = \frac{3}{8}$

$P(A)\,P(B) = \frac{\cancel{6}^{3}}{8}\times\frac{\cancel{4}}{\cancel{8}_{2}} = \frac{3}{8} = P(A\cap B)$. Therefore A & B <u>are independent</u>.

3 events A, B, C

If
$$\left.\begin{array}{l} P(A,B) = P(A)P(B) \\ P(A,C) = P(A)P(C) \\ P(B,C) = P(B)P(C) \end{array}\right\}, \text{ then } A, B, C \text{ are pairwise independent (PI).}$$

If, in addition $P(A,B,C) = P(A)P(B)P(C)$, then A, B, C are mutually indepedent (MI).

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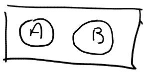
3 H in 4 tosses of a fair coin. — S was simple. P via counting.

→ 3H in 4 tosses of a biased coin?

① Independence $\boxed{P(A \cap B) = P(A)P(B)}$

② Mutually Exclusive events (ME)

- If A has occured, then B definitely did not and cannot occur.
- H and T on coin toss are ME.
- 1, 2, 3, 4, 5, 6 on die roll are ME.



ME

If A & B are ME, then $A \cap B = \emptyset$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

For ME events,
$$\boxed{P(A \cup B) = P(A) + P(B)}$$

Are mutually exclusive events independent? Absolutely not!! They are dependent.

Bernoulli Experiment

Coin toss.
Output is either 0 or 1
T or H
fail or pass

Only 2 outcomes.

Bernoulli Process

A sequence of Bernoulli experiments.
coin tosses using the same coin.
P(H) is same for each toss.

Binomial Experiment

A Bernoulli process of n coin tosses and recording the number of H's.
① Toss a coin n times.
② Record/count the number of heads.

$S = \{0, 1, 2, \cdots, n\}$

3H in 4 tosses of a biased coin. ⟹ Binomial exp. with $n = 4$.
$P(H) = p$.

Tossing a coin 4 times
↳ $S_1 = \Big\{$

$P(H) = p$