Conditional Probability

P(rain tomorrow)

P(rain tomorrow given that the Met department has forecasted rain for tomorrow)

P(Pakistan winning the Cricket match) = 0.6

vs

$$\underbrace{P(\text{Pakistan wining} \mid \underbrace{\text{Babar Azam is injured}}_{\text{given}})}_{\text{conditional probability.}} = 0.4$$

S = green rectangle

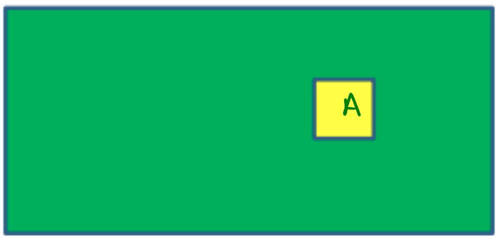
A = yellow square

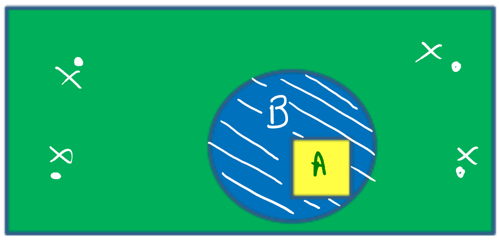
$$P(A) = \frac{area(A)}{area(S)}$$

B = blue circle

P(A) = area (A) / area(S)

P(B) = area(circle) / area(rectangle) = area(B)/area(S)



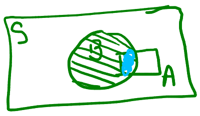


P(A|B) = P(pt. lies in square A given that selected point lies in the circle)

$$= \frac{area(A \cap B)}{area(B)}$$

$$= \frac{area(A \cap B) / area(S)}{area(B) / area(S)}$$

$$= \frac{P(A \cap B)}{P(B)}$$

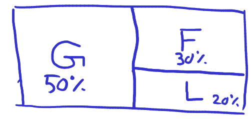


$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

Q. Suppose that out of all fans manufactured in Punjab,

20% are manufactured in Lahore
50% ______ Gujrat
30% ______ Faisalabad



Partition of S into G, F & L.

$G \cup F \cup L = S$

$G \cap F = \emptyset$
$G \cap L = \emptyset$
$F \cap L = \emptyset$

5% are manufactured in Lahore are defective
3% ______ Gujrat ______
4% ______ Faisalabad ______

D = a fan manufactured in Punjab is defective
L = fan was ______ Lahore
G = ______ Gujrat
F = ______ Faisalabad

(i) What is $P(L), P(G)$ & $P(F)$?

$P(L) = 20\% = 0.2$
$P(G) = 50\% = 0.5$
$P(F) = 30\% = 0.3$

(ii) $P(D|G)$?

P(a fan is defective given that it was manufactured in Gujrat) $= 3\% = 0.03$

(iii) $P(D)$? P(fan is defective) = ?

(iv) $P(G|D)$?

P(a fan was manufactured in Gujrat given that it was defective) = ?

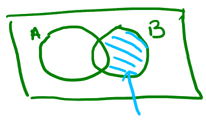
Properties of Conditional Probability

Normal

(1) $P(A^c) = 1 - P(A)$

Conditional

$$P(A^c|B) = \frac{P(A^c \cap B)}{P(B)}$$



$A^c \cap B$

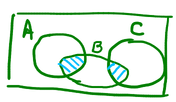
$$= \frac{P(B) - P(A \cap B)}{P(B)}$$

$$= 1 - \frac{P(A \cap B)}{P(B)} = 1 - P(A|B)$$

(2) For disjoint events A and C

$P(A \cup C) = P(A) + P(C)$

$$P(A \cup C|B) = \frac{P((A \cup C) \cap B)}{P(B)}$$



$$= \frac{P(A \cap B) + P(B \cap C)}{P(B)}$$

$$= P(A|B) + P(C|B)$$

③ $P(A|B) = \dfrac{P(A\cap B)}{P(B)} \Rightarrow P(A\cap B) = P(A|B)\,P(B)$

Notice the difference

$P(B|A) = \dfrac{P(A\cap B)}{P(A)} \Rightarrow P(A\cap B) = P(B|A)\,P(A)$

$$P(A\cap \underbrace{B\cap C}) = P(A|B\cap C)\,\underbrace{P(B\cap C)}$$
$$= P(A|B\cap C)\;P(B|C)\,P(C)$$

$$P(A\cap B\cap C) = P(C|A\cap B)\,\underbrace{P(A\cap B)}$$
$$= P(C|A\cap B)\;P(B|A)\,P(A)$$

④ If A & B are independent, then

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$$

⑤ If A & B are mutually exclusive, then

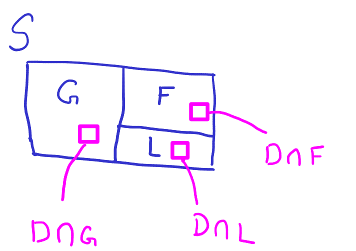
$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = \frac{0}{P(B)} = 0$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{0}{P(A)} = 0$$

Q. ...continued

(iii) $P(D) = ?$

P(a fan is defective)



$$D = \underbrace{(D\cap L)} \cup \underbrace{(D\cap G)} \cup \underbrace{(D\cap F)}$$

Mutually exclusive

$$P(D) = P(D\cap L) + P(D\cap G) + P(D\cap F)$$

$$= \underbrace{P(D|L)}_{0.05}\underbrace{P(L)}_{0.2} + \underbrace{P(D|G)}_{0.03}\underbrace{P(G)}_{0.5} + \underbrace{P(D|F)}_{0.04}\underbrace{P(F)}_{0.3}$$

$$= 0.037$$

Theorem of Total Probability

If events $B_1, B_2, \ldots$ form a partition of S, then for any event A

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots$$

Total

$$= P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots$$

(iv) $P(G \mid D) = ?$

P(fan was manufactured in Gujrat given that it was defective)

$$P(G \mid D) = \frac{P(G \cap D)}{P(D)} = \frac{P(D \mid G)P(G)}{P(D)}$$

TTP

Theorem

Bayes' Rule

$$P(A \mid B) = \frac{P(B \mid A)\,P(A)}{P(B)}$$

$$= \frac{0.03 \times 0.5}{0.037} = 0.4054$$

Combining everything together

$$P(B_k \mid A) = \frac{P(A \mid B_k)\; P(B_k)}{P(A)} = \frac{P(A \mid B_k)\,P(B_k)}{\sum_i P(A \mid B_i)\,P(B_i)}$$