Lecture 2

Assigning Probabilities via Size – I

In some sense the probability of an event should be related to its "size". The larger the event the larger its probability, by part (v) of Theorem 1.2.1. That is, we may try the formula

$$\mathbb{P}(A) \;=\; \frac{\text{size of } A}{\text{size of } S}, \quad \text{for any event } A,$$

to define the function $\mathbb{P}$. Any 'reasonable' interpretation of the word "size" leads to a probability assignment technique that does not violate the three axioms, (2.1), (2.2), (2.3), of $\mathbb{P}$ listed in the last Lecture. For instance, when the sample space S is a finite set, we may think of the "size" of A to stand for the number of elements A has. In this case we may define

$$\boxed{\mathbb{P}(A) \;=\; \frac{\text{Number of elements in } A}{\text{Number of elements in } S} \;=\; \frac{\text{Number of favorable}}{\text{Number of plausible}}.}$$

With this definition, S is called a simple sample space and the outcomes are called equally likely (or equilikely). This assignment technique deals with counting elements of sets, which may or may not be simple. So, we start off with a refresher on counting skills.

2.1 Counting & The Multiplication Principle

Example - 2.1.1 - A fair die is rolled three times. What is the probability that all three faces will be different?

How many elements does the sample space S have? Each element of S looks like a tripple $(-,-,-)$, where the empty spots are filled with numbers ranging from $1, 2, \cdots, 6$. The trick in counting deals with an intuitive fact:

> "If a task is completed in stages (say two stages), so that the first stage can be completed in m ways and the second stage can be completed in n ways then the whole task can be completed in mn ways."

This is called the multiplication principle and can be viewed (Figure 2.1) as the number of complete branches of a tree.

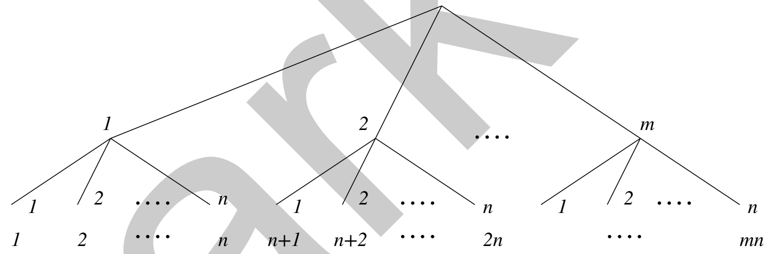

Figure 2.1: Multiplication Principle in Tree Diagram Form.

For our example, the elements of S are built in three stages, therefore ther are $6 \times 6 \times 6 = 216$ elements. Here the use of the multiplication principle is evident. The sample space is obviously too big to write all its elements. The statement that the "die is fair", means that every element of S has the same probability of occurring, making the sample space simple.

Our event A contains all those elements of S for which all faces are different. How many elements does A have? A typical element of A looks like $(-,-,-)$, where the dashes will occupy the face values of the three rolls that are different. Note that the first spot can be filled in any one of 6 ways, the second spot can then be filled in any one of 5 ways, since the second roll must not have what was already seen on the first roll. Similarly, the third roll must be different from the first two, which leaves only 4 choices. Thus by the multiplication principle, A consists of $6 \times 5 \times 4 = 120$ elements. So, the probability of A is

$$\mathbb{P}(A) = \frac{\text{number of elements in } A}{\text{number of elements in } S} = \frac{120}{216} = 0.55555.$$

Remark - 2.1.1 - **(Permutations)** There are n people in a room, from which we need to select 3 people, one of which will serve as the president, another as the secretary and the third as the treasurer. How many ways can we make such a selection?

Any such selection is called a permutation (or ordered arrangement) of size 3. We can use the multiplication principle to count the total number of permutations of size 3 taken from n people. The president can be chosen in n ways, then the secretary can be picked from the remaining $n-1$ people in $n-1$ ways and then in $n-2$ ways we can pick a treasurer from the remaining that many people. Hence, the total number of ways of selecting the three officers is

$$n \times (n-1) \times (n-2) \;=\; \frac{n!}{(n-3)!}.$$

In general, a permutation of k distinct objects is an ordered arrangement of the objects. If k objects are selected in order from a collection of n objects, the total number of permutations is

$$\boxed{{}_nP_k \;:=\; \frac{n!}{(n-k)!}, \qquad k = 1, 2, \cdots, n.}$$

Example - 2.1.2 - **(Birthday matching problem)** If three people are in a room, what is the probability that no two share the same birthday?

Assuming a year consists of 365 days, the sample space consists of 365^3 number of elements, where an element is a triple $(-, -, -)$ listing the birthdays of the three people. The event of interest, A, consists of those triples in which the birthdays of the three individuals are different. The first place could be any one of 365 numbers, and the second place therefore could be any one of the remaining 364 numbers and then the last position could be any one of the remaining 363 numbers. Hence, A consists of $365 \times 364 \times 363$ elements. Assuming that the elements of the sample space are equally likely, the probability of A is $\frac{365\times364\times363}{365^3} = 0.9918$.

Example - 2.1.3 - **(Problem of Chevalier de la Mèrè)** Prove that at least one 6 in four rolls of a fair die is more likely than at least one double six in 24 rolls of a pair of fair dice. (It is claimed that Chevalier de la Mèrè knew about this fact by his extensive gambling experience.)

First note that the probability of no sixes in the four rolls is

$$\frac{5^4}{6^4}.$$

Therefore, the probability of at least one 6 in four rolls is

$$1 - \frac{5^4}{6^4}.$$

Similarly, the probability of at least one double six in 24 rolls is

$$1 - \frac{35^{24}}{36^{24}}.$$

By taking the log, we can see the observation attributed to de la Mèrè. Indeed,

$$24 \log(35/36) = -0.6761 \;>\; -0.729 = 4 \log(5/6).$$

Example - 2.1.4 - **(The secretary problem)** A secretary types n letters addressed to n different people. Then he types n addresses on envelopes for the same n people. However, he inadvertently inserts the letters into the envelopes randomly ignoring who's letter is put into who's envelope. We would like to know the probability of the event, A, that at least one of the letters is correctly put into its own envelope. Consider a simple case first. Suppose there are only $n = 2$ letters with two envelopes addressed to Tom and Dick. There are only two possibilities, $S = \{a, b\}$, where

a: stands for Tom gets Dick's letter and Dick gets Tom's letter,
b: stands for Tom gets Tom's letter and Dick gets Dick's letter.

Our event is $A = \{b\}$. When letters are put in the envelopes blindly (randomly), the outcomes, a, b, become equilikely, making S a simple sample space. Therefore,

$$\mathbb{P}(A) = \frac{\text{Number of elements in } A}{\text{Number of elements in } S} = \tfrac{1}{2} = 0.5.$$

For higher values of n, and using some advanced counting techniques, the general formula for $\mathbb{P}(A)$ turns out to be (see [37] for instance)

$$\mathbb{P}(A) \;=\; 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!} \;=\; \sum_{j=1}^{n} (-1)^{j+1}\frac{1}{j!} \;\approx\; 1 - e^{-1} = 0.632.$$

This example shows that we need to learn some accurate ways of counting elements of a set.

2.2 Combinations & Some Card Games

The second trick of counting involves binomial coefficients:

$$\boxed{\binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \qquad k = 0, 1, 2, \cdots, n.}$$

It gives the number of ways a group of k items can be picked out (all at once without regard to order) from a group of n items. Any such selection of a group, since order is not involved, is called a combination. This leads to the binomial series,

$$(x+y)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} x^k\, y^{n-k}, \qquad \text{for any positive integer } n.$$

The following examples show how to use combinations and the multiplication principle together in the same counting problem.

Example - 2.2.1 - **(Drawing without replacement)** A basket contains 8 apples and 9 oranges all mixed up. We reach in, without looking, draw three items all at once.

- (i) What is the total number of ways we can draw 3 items?
- (ii) What is the total number of ways we can draw 3 items such that 2 are apples and one is an orange?
- (iii) What is the probability that we will get 2 apples and one orange?

The sample space, S, consists of $\binom{8+9}{3} = 680$ combinations, which answers (i). Each combination is a possible draw of three items from the basket. Since drawing is done blindly and the basket is shaken well, each draw has the same chance, it is safe to say that the sample space is simple.

The event of interest, A, consists of all those combinations in which there are 2 apples and one orange. Imagine drawing the three items a bit more carefully so that we are sure that the resulting hand will have 2 apples and one orange. This can be done in two stages. In stage one the drawn combination is made to contain two apples — which can be done in $\binom{8}{2} = 28$ ways. In the second stage the drawn combination is made to contain an orange — which can be done in $\binom{9}{1} = 9$ ways. Therefore, by the multiplication principle, the drawn hand can be made up, so that it will have 2 apples and one orange, in $28 \times 9 = 252$ ways. This answers (ii). Since the sample space is simple, this gives

$$\mathbb{P}(A) = \frac{\text{number of elements in } A}{\text{number of elements in } S} = \frac{\binom{8}{2}\binom{9}{1}}{\binom{8+9}{3}} = \frac{252}{680} = 0.370588.$$

Example - 2.2.2 - **(Hypergeometric model)** An urn contains G number of good apples and B number of bad apples. We reach in and draw n apples all at once. What is the probability that we will have drawn k number of good apples?

It should be clear that if k is negative or greater than n or greater than G then the probability of such an event must be zero. So, consider the more relevant case when $k = 0, 1, 2, \cdots, \min\{G, n\}$. In this case there are $\binom{G+B}{n}$ ways of drawing n apples from the urn. That is, our sample space consists of $\binom{G+B}{n}$ elements. The event, A_k consists of those elements which have exactly k number of good apples. The number of elements in A_k is

$$\binom{G}{k} \times \binom{B}{n-k}.$$

This is because there are $\binom{G}{k}$ number of ways we can ensure that our drawn hand has k number of good apples. The rest of the drawn apples should be all bad, but that can be accomplished in $\binom{B}{n-k}$ ways. The multiplication principle then gives the total number of elements of A_k. Therefore,

$$\mathbb{P}(\text{exactly } k \text{ good apples}) = \frac{\binom{G}{k}\binom{B}{n-k}}{\binom{G+B}{n}}, \qquad k = 0, 1, 2, \cdots, \min\{G, n\}.$$

This is often called the hypergeometric model.

Example - 2.2.3 - **(Card games)** Consider a fully shuffled standard deck of 52 playing cards. We will compute the probability of receiving two pairs while randomly drawing a hand of five cards.

- Since the deck is fully shuffled, and the drawing is taking place at random, (i.e., any hand of 5 cards has equal chance of being selected) the sample space is simple.
- The sample space is quite large, consisting of

$$\binom{52}{5} = 2,598,960$$

elements. Each element of S is a hand of 5 cards, which may be viewed as a set of five cards $\{-,-,-,-,-\}$. So, each hand (consisting of 5 cards) has the (equal) probability of being selected is $\frac{1}{\binom{52}{5}} = 0.000000384$.

- Our event, A, consists of all those hands (elements of S) which have two pairs. I claim that A has

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$$

elements (hands of five cards). To see why, proceed systematically, by invoking the multiplication principle. There are thirteen groups, labeled A, 2, $\cdots$, 10, J, Q and K. First there are $\binom{13}{2}$ ways to choose the two groups (which will give the two pairs). Each group has four cards in it (from the four suits). Each pair is chosen in $\binom{4}{2}$ ways. After the pairs are chosen, throw away the remaining cards of those two groups (to avoid making a triple inadvertently). The fifth card is then chosen in $\binom{44}{1}$ way from the remaining 44 cards.

- So, the probability of the event is $\mathbb{P}(A) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}} \approx 0.047539$.

Remark - 2.2.1 - **(Multinomial coefficient)** An extension of binomial coefficient is known as multinomial coefficient. Here is the idea. We roll a die 10 times and record their outcomes. The total number of possible outcomes is 6^{10}. To count the number of outcomes that have face 6 repeated two times, face 1 repeated three times, and face 2 repeated five times, using binomial coefficients and the multiplication principle, we have

$$\underbrace{\binom{10}{2}}_{\text{places where face 6 appears}} \times \underbrace{\binom{10-2}{3}}_{\text{places where face 1 appears}} \times \underbrace{\binom{10-2-3}{5}}_{\text{places where face 2 appears}} = \frac{10!}{2!\,3!\,5!}.$$

This is called a multinomial coefficient. In general, the expression

$$\frac{m!}{k_1!\,k_2!\cdots k_n!}, \qquad k_1 + k_2 + \cdots + k_n = m,$$

is called a multinomial coefficient and written as $\binom{m}{k_1, k_2, \cdots, k_n}$. When $n = 2$ it reduces to a binomial coefficient, written as $\binom{m}{k_1}$ instead of $\binom{m}{k_1, k_2}$ since $k_1 + k_2 = m$ is fixed. The multinomial expansion formula is a simple extension of the binomial series:

$$(x_1 + x_2 + \cdots + x_n)^m = \sum_{(k_1, k_2, \cdots, k_n)} \binom{m}{k_1, k_2, \cdots, k_n} x_1^{k_1}\, x_2^{k_2} \cdots x_n^{k_n},$$

where the sum is over all n-tuples of nonnegative integers with $\sum_{i=1}^{n} k_i = m$.

Example - 2.2.4 - **(Multinomial probabilities)** From a well shuffled standard deck of cards a card is drawn and its suit is noted, then the card is put back into the deck and the deck is reshuffled. This is repeated ten times. Find the probabilities of the following events.

- (i) Observing the sequence $\diamondsuit\heartsuit\diamondsuit\clubsuit\clubsuit\diamondsuit\spadesuit\heartsuit\heartsuit\diamondsuit$ in this order.
- (ii) Observing 4 diamonds, 3 hearts, 2 clubs and 1 spade disregarding order.

Each draw has four varieties. (Although there are thirteen cards in each suit, for our purpose they are indistinguishable, and hence we consider the deck consisting of only four cards representing the four suits.) By the multiplication principle, there are 4^{10} different ways the draw of ten cards can occur. There is exactly one way to have the draw in the order $\diamondsuit\heartsuit\diamondsuit\clubsuit\clubsuit\diamondsuit\spadesuit\heartsuit\heartsuit\diamondsuit$. Therefore,

$$\mathbb{P}(\diamondsuit\heartsuit\diamondsuit\clubsuit\clubsuit\diamondsuit\spadesuit\heartsuit\heartsuit\diamondsuit) \;=\; \frac{1}{4^{10}}.$$

(ii) There are exactly $\binom{10}{4,3,2,1}$ ways the ten cards consisting of $\diamondsuit\heartsuit\diamondsuit\clubsuit\clubsuit\diamondsuit\spadesuit\heartsuit\heartsuit\diamondsuit$ can be ordered. Therefore,

$$\mathbb{P}(\text{4 diamonds, 3 hearts, 2 clubs and 1 spade regardless of order}) \;=\; \frac{10!}{4!\;3!\;2!\;1!}\,\frac{1}{4^{10}}.$$

Remark - 2.2.2 - **(From one dimension to zero dimension — Poisson model)** The size of a set does not have to be its number of elements. This extension is needed especially if the sample space S consists of infinitely many elements. Let $S = \{0, 1, 2, \cdots\}$. Break the interval $[0, 1]$ into subintervals whose sizes are $(e^{-13}\frac{(13)^0}{0!})$, $(e^{-13}\frac{(13)^1}{1!})$, $(e^{-13}\frac{(13)^2}{2!})$, $(e^{-13}\frac{(13)^3}{3!})$, et cetera. By the exponential series these interval lengths do add up to one,

$$(e^{-13}\frac{(13)^0}{0!}) + (e^{-13}\frac{(13)^1}{1!}) + (e^{-13}\frac{(13)^2}{2!}) + (e^{-13}\frac{(13)^3}{3!}) + \cdots \;=\; e^{-13}e^{13} \;=\; 1.$$

Assign the element $k \in S$ the length of the k-th interval, as its probability. This gives a well known probability assignment model, called a Poisson model. Radio activity counters, and predictions of number of telephone calls, use this kind of models. This is an example of non-equilikely probability space.

2.3 Exercises

Exercise - 2.3.1 - *Let $S = \{a, b, c, d\}$ be a sample space with the power set as the class of events. Let $\mathbb{P}(\{a\}) = \mathbb{P}(\{b\}) = \frac{1}{4}$. What should be the value of $\mathbb{P}(\{c\})$ so that the sample space is simple? If we take $\mathbb{P}(\{c\}) = \frac{1}{12}$, would the sample space be simple?*

Exercise - 2.3.2 - *Write the sample space for the secretary problem when $n = 4$ letters are sent. Verify that the probability of at least one correct letter is 0.625.*

Exercise - 2.3.3 - *A number is selected "blindly" from the first 100 positive integers. What is the probability that the selected number is divisible by three?*

Exercise - 2.3.4 - *If a room consists of 4 unrelated people what is the probability that at least two of them share the same birthday?*

Exercise - 2.3.5 - *A box contains 30 mangoes, 10 of which are rotten and the rest are OK. If 4 mangoes are drawn blindly from the box all at once, what is the probability that one mango will be rotten and the rest of them are OK?*

Exercise - 2.3.6 - *If six fair dice are rolled, find the probability that all faces will be different.*

Exercise - 2.3.7 - *A person has 5 shirts, 6 trousers, and 7 pairs of socks.*

- *(i) In how many ways can he dress up (using one shirt, one trouser and one pair of socks)?*
- *(ii) In how many ways can he dress up without wearing any socks?*
- *(iii) If four of the trousers need repairs, and he blindly selects a trouser, what is the likelihood that he will be wearing a trouser which needs repair?*

Exercise - 2.3.8 - *Compute the probability that a group of five cards drawn at random from a standard deck of 52 cards will contain (i) $B = \{$exactly one pair$\}$, (ii) $C = \{$a full house$\} = \{$ one pair and one triple $\}$, (iii) $D = \{$a flush$\} = \{$all five from the same suit$\}$. (iv) $E = \{$a royal flush$\} = \{$Ace, King, Queen, Jack, Ten and all from the same suit$\}$.*
By the way, the order/ranking of poker hands is as follows.

- *0: None of the following.*
- *1: One pair; one pair of equal ranks in the five cards.*
- *2: Two pairs; two pairs of equal ranks in the five cards.*
- *3: Three of a kind; three equal ranks in the five cards.*
- *4: Straight; five cards in sequential order without gaps.*
- *5: Flush; five cards with the same suit, (but not straight flush or royal flush as stated below).*
- *6: Full house; a pair and three of a kind of different rank.*
- *7: Four of a kind; four equal ranks in the five cards.*
- *8: Straight flush; a flush and a straight.*
- *9: Royal flush; Ace, King, Queen, Jack, Ten and all from the same suit.*

Exercise - 2.3.9 - *Find the probability of getting at least one double six in six rolls of a pair of fair dice.*

Exercise - 2.3.10 - *A fair die is rolled six times. Find the probability of the event that not all six faces were observed.*

Exercise - 2.3.11 - *There are seven people in a party. Assume each one of them could have been born in any one of the 365 days of a year. In how many possible ways they could all have been born?*

Exercise - 2.3.12 - *There are seven people in a party. Assume each one of them could have been born in any one of the 365 days of a year with equal chance. What is the probability that they all have different birth days.*