# Lecture 8

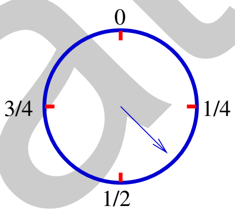
# Special Continuous R.Vs

Continuous distributions are mathematical models of real-life random experiments which may, at best, only approximately represent the actual probabilistic behavior of the random experiment. We try to have some justifications while linking an actual random experiment with a particular distribution. These approximate models are quite useful in just about all fields of knowledge. Even when the model cannot be justified from a practical point of view, it can still be used to better understand the consequences of different possible scenarios. In the following we list a few of these model distributions along with their important properties.

## 8.1 Some Famous Continuous Name Brands

Here we collect a few continuous random variables. In the end of the lecture exercises a few more examples are provided. All these random variables have wide applications.

**Example - 8.1.1** - **(Uniform random variables)** Consider a spinner which does not have any preferential stopping region. How should we model this phenomenon? If $X$ represents the point where the pointer stopped, $X$ "selects a point at random" from the interval $[0,1]$. This is modeled by a uniform density. A random variable $X$ is called a uniform random variable on an interval $[\alpha, \beta]$ if the probability density function of $X$ is given by




$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha < x < \beta, \\ 0 & \text{otherwise.} \end{cases}$$

For our spinner, $\alpha = 0$, $\beta = 1$. The numbers $\alpha, \beta$ are called its parameters that we can adjust to represent various applications. We will denote this random variable





by $X \sim Uniform(\alpha, \beta)$. The distribution function (cdf) of $X$ is

$$F(t) = \begin{cases} 0 & \text{if } t < \alpha, \\ \int_\alpha^t f(x)dx & \text{if } \alpha \le t \le \beta, \\ 1 & \text{if } \beta < t, \end{cases} = \begin{cases} 0 & \text{if } t < \alpha, \\ \frac{t-\alpha}{\beta-\alpha} & \text{if } \alpha \le t \le \beta, \\ 1 & \text{if } \beta < t. \end{cases}$$

Two examples of this density are shown in Figure 8.1. $X$ represents the random

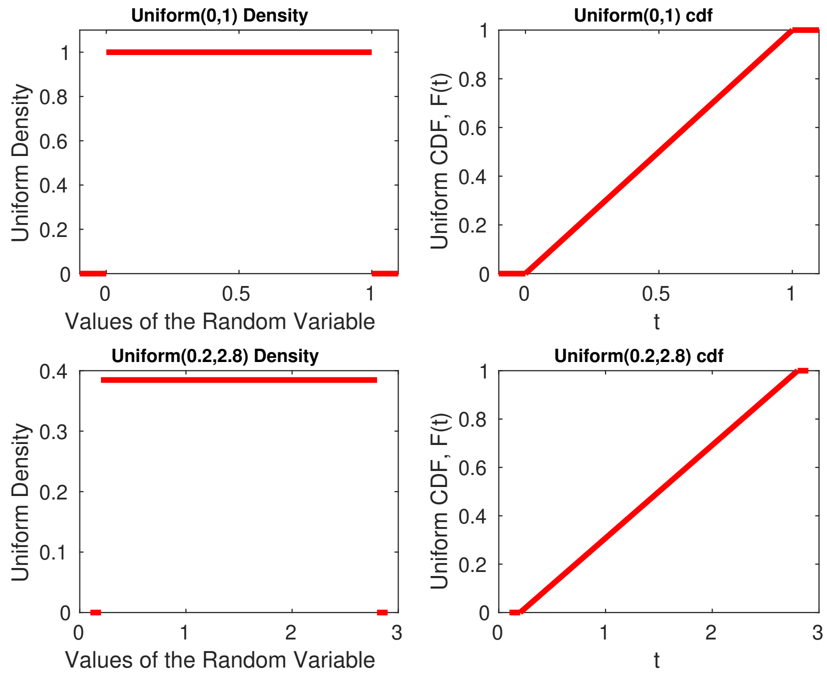



Figure 8.1: Two Uniform Densities with their CDFs.

experiment of "blindly" selecting a point from the interval $[\alpha, \beta]$. Note $\frac{dF(t)}{dt} = f(t)$ for all $t \in \mathbb{R}$ except at two points, $\alpha, \beta$, where it does not matter since their lengths (and hence probabilities) add up to zero. The special case of $\alpha = 0$, $\beta = 1$ is the back bone of simulation and Monte Carlo methods. Most computers have algorithms to select points "at random" from $[0,1]$ and the resulting numbers are called "pseudo-random numbers".

**Example - 8.1.2** - **(Beta random variables)** A probability model for studying percentages is a generalization of $Uniform(0,1)$, called the beta distribution. The general form of its density is

$$f(x) = \begin{cases} \frac{\Gamma(a+b)}{\Gamma(a)\ \Gamma(b)} x^{a-1}(1-x)^{b-1}, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The constants $a, b > 0$ are its parameters. Its notation is $X \sim Beta(a, b)$. For $a = b = 1$ we get the $Uniform(0, 1)$ distribution. For the the special case of $b = 2$, for example, its cdf is

$$\begin{aligned} F(t) &= \frac{\Gamma(a+2)}{\Gamma(a)} \int_0^t x^{a-1}(1-x)\,dx \;=\; a(a+1)\int_0^t x^{a-1}(1-x)\,dx \\ &= a(a+1)\left(\frac{x^a}{a} - \frac{x^{a+1}}{a+1}\right)\Bigg|_{x=0}^{x=t} \;=\; a(a+1)\left(\frac{t^a}{a} - \frac{t^{a+1}}{a+1}\right) \\ &= t^a\,(a+1-at)\,, \qquad 0 < t < 1. \end{aligned}$$

$F(t) = 0$ for $t \le 0$ and $F(t) = 1$ for $t \ge 1$. For arbitrary values of $a, b$, there is no nice closed form expression for $F(t)$.

**Example - 8.1.3** - **(Exponential rvs — modeling radio activity)** The time it takes for a Carbon-14 atom to decay is used to date ancient carbonaceous artifacts. The half-life of the unstable Carbon-14 isotope is roughly around $5,730$ years. That is, if $C$ amount of carbon-14 material is left to decay naturally, after $5,730$ years $\frac{C}{2}$ amount will be left. How long will it take to have only $\frac{C}{4}$, or $\frac{C}{8}$ or $\frac{C}{16}$ amount to remain? More generally, if $F(t) \in (0, 1)$ is the percentage of the amount decayed away and $t$ is the needed amount of time, then this $F(t)$ is a CDF that is well approximated by an exponential distribution, $1 - e^{-\lambda t}$, with density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is a parameter representing the rate of decay. To see the link between the rate of decay and half-life, we must have

$$\frac{1}{2} \;=\; F(5730) \;=\; \int_0^{5730} \lambda e^{-\lambda x}\,dx \;=\; 1 - e^{-\lambda 5730}.$$

Solving for $\lambda$ we get

$$\lambda \;=\; \frac{\ln 2}{\text{half-life}} \;=\; 0.00012097.$$

In statistical literature the half-life is called the median of the density. We denote an exponential random variable by $X \sim Exp(\lambda)$. In this context, $X$ can be interpreted as the waiting time until a typical (or randomly picked) Carbon-14 isotope will decay. Two examples of this density and cdf are shown in Figure 8.2.

To see how this random variable is used for carbon dating, a bone sample of an Egyptian mummy was found to have 60% of the total Carbon-14. How old is the mummy? We need to solve for $t$,

$$1 - 0.60 \;=\; 0.40 \;=\; F(t) \;=\; 1 - e^{-0.00012097t}.$$

This gives that the mummy is $t = 4,222$ years old.





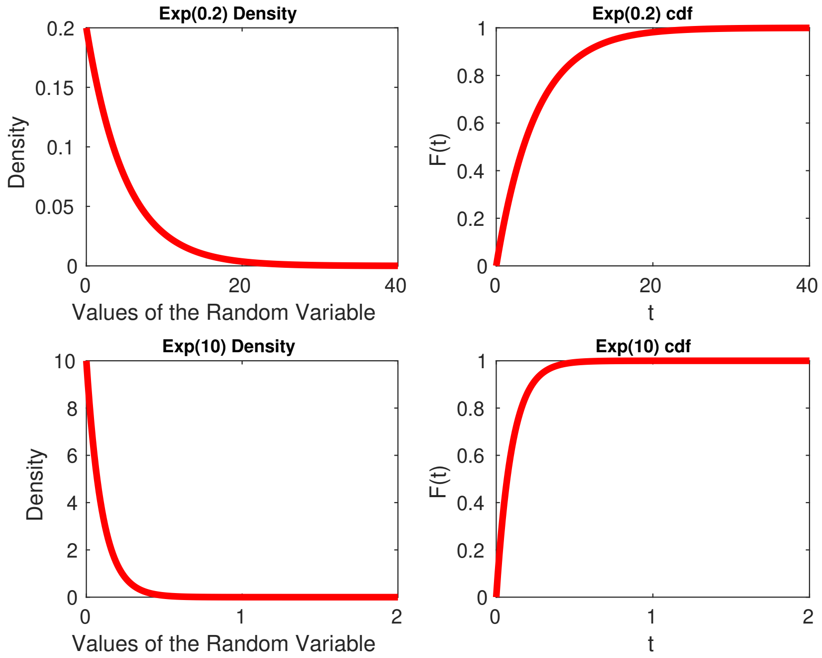



Figure 8.2: Two Exponential Densities with their CDFs.

**Example - 8.1.4** - **(Gamma and chi square densities)** Cumulative life lengths of several resistors, (or appliances) is modeled by extending the exponential density as follows. Wholesale dealers use such models to predict the demand, or warranty durations, for their products. A continuous random variable $X$, with density function

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{otherwise,} \end{cases}$$

is called a gamma random variable. We denote this by $X \sim G(\lambda, \alpha)$ where $\lambda, \alpha > 0$ are its parameters. Two examples of this density are shown in the first row of Figure 8.3. Note that when $\alpha = 1$, $G(\lambda, 1) \sim Exp(\lambda)$.

Some gamma densities appear so often in Statistics that they are given a special name. If $X \sim G(\frac{1}{2}, \frac{n}{2})$ (i.e., $\lambda = \frac{1}{2}$ and $\alpha = \frac{n}{2}$ where $n$ is a positive integer) then $X$ is also called a chi square random variable with $n$ degrees of freedom and is also denoted as $X \sim \chi^2_{(n)}$. Two examples of this density are shown in the second row of Figure 8.3.

**Example - 8.1.5** - **(Normal random variables — modeling measurement errors)** All measurements are contaminated with errors. The cumulative effect of several sources of measurement errors is modeled by normal distributions. An

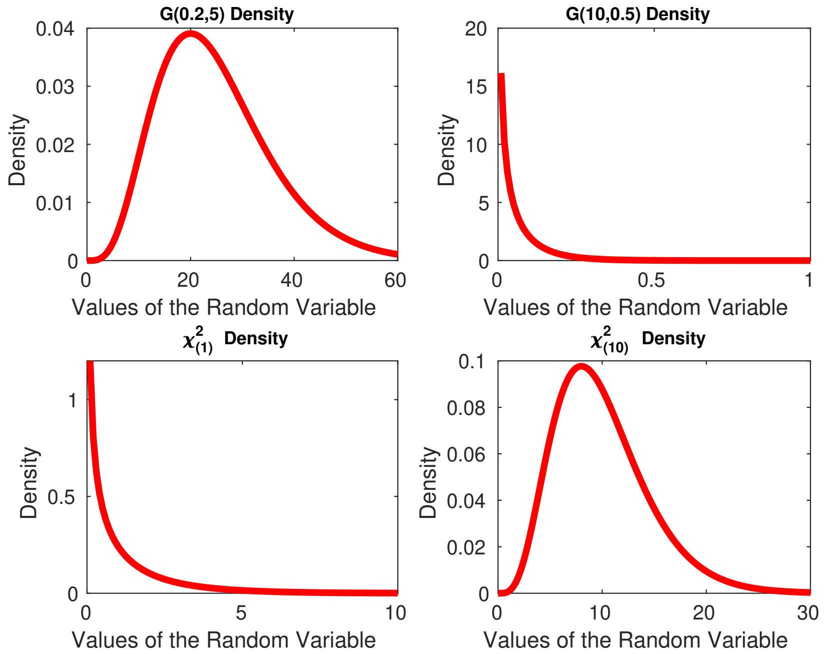



Figure 8.3: Two Gamma and Two Chi Square Densities.

example of this was seen in the beginning of the last lecture while modeling the fill weights of lemonade cans. A continuous random variable, $X$, having the probability density,

$$f(x) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2b^2}(x-a)^2}, \qquad x \in \mathbb{R},$$

is called a normal or a Gaussian[1] random variable with parameters $a, b$, where $a \in (-\infty, \infty)$ specifies the location of the hump and $b > 0$ specifies the distance of the point of inflection form the location of the hump. We denote this by $X \sim N(a, b^2)$. Further meanings of $a, b^2$ will become clear later. Three normal densities are shown in Figure 8.4. Luckily, the cdf, $\mathbb{P}(X \le t)$, of all normal random variables is linked to one very special case, called the standard normal random variable.

Recall from Section 7.3, $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$. This integral gives rise to a very special normal density. A continuous random variable, $Z$, with a probability

[1] Normal distributions were discovered by A. DeMoivre (1667-1754) in 1733 as an approximation of the binomial distribution $B(n, \frac{1}{2})$ for large values of $n$. Later, Laplace (1749-1827) extended DeMoivre's result by approximating $B(n, p)$ for any $0 < p < 1$. The German mathematician, Carl F. Gauss (1777-1855), apparently rediscovered it while modeling the orbits of celestial bodies when measurements had errors, explaining why this distribution is also known by his name.





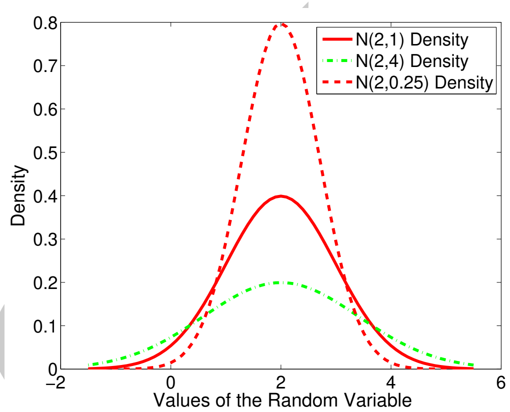



Figure 8.4: Three Normal Densities.

density function and distribution function,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad x \in \mathbb{R}, \qquad F(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx,$$

is called a standard normal random variable. We denote this random variable by $Z \sim N(0,1)$. This density is shown in Figure 8.5. Some books and we will also

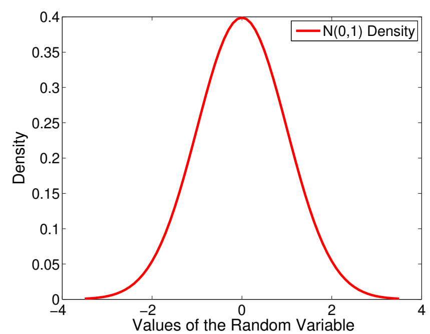



Figure 8.5: The Standard Normal Density.

denote the cdf, $F(t)$ of $Z$ by $\Phi(t)$ and its density by $\phi(x)$. A short table for $\Phi(t)$ is given below.

| $t$ | $-3.0$ | $-2.5$ | $-2.0$ | $-1.96$ | $-1.65$ | $-1.47$ | $-1.28$ | $-1.0$ | $-0.5$ |
|---|---|---|---|---|---|---|---|---|---|
| $\Phi(t)$ | .0013 | .0062 | .0228 | .0250 | .0496 | .0708 | .1003 | .1587 | .3085 |

The main justification of why a normal random variable is related to the cumulative effect of measurement errors is provided by a theorem called the central limit theorem to be proved later.

When $X \sim N(a, b^2)$ and $Z \sim N(0,1)$, one useful link between the distributions of these two random variables is

$$\mathbb{P}(X \le t) \;=\; \mathbb{P}\left(\frac{X-a}{b} \le \frac{t-a}{b}\right) \;=\; \mathbb{P}\left(Z \le \frac{t-a}{b}\right).$$

That is, $\frac{X-a}{b} \sim Z$. Hence the distribution $\Phi(t)$ of $Z$ may be used to find the distribution of any other normal random variable. This is called the z-transformation link. For instance, from Example 7.1.1 when $X \sim (20, 0.25)$ is the fill weight of a randomly chosen can, the chances of it having less than 18.5 oz are (by using the above short table)

$$\mathbb{P}(X \le 18.5) \;=\; \mathbb{P}\left(Z \le \frac{18.5-20}{0.5}\right) \;=\; \mathbb{P}(Z \le -3) \;=\; 0.0013.$$

**Example - 8.1.6** - **(Cauchy random variables — modeling symmetric heavy tails phenomenon)** The function

$$f(x) \;=\; \frac{b}{\pi\,(b^2 + (x-a)^2)}, \qquad -\infty < x < \infty$$

is nonnegative and the total area under it is one. This makes it a density having parameters $a \in \mathbb{R}$, and $b > 0$. A random variable having this density is called a Cauchy random variable. We will denote this by $X \sim Cauchy(a, b)$. It is not difficult to see that the cdf of $X$ is

$$F(t) \;=\; \frac{1}{2} + \frac{1}{\pi}\tan^{-1}((t-a)/b), \qquad -\infty < t < \infty.$$

Figure 8.6 shows the comparison of the standard normal $N(0,1)$ and $Cauchy(0,1)$ densities. Note the difference in the tails. Cauchy densities serve as models where the measurements can take large values in both extremes.

For instance, consider the tunnel Exercise 3.3.15. If $\Theta$ is taken to be the randomly selected angle in $[0, \pi]$ and $X = \cot(\Theta)$ is the resulting value as shown in the figure associated to that exercise, then $\mathbb{P}(X > t) = \frac{1}{\pi}\tan^{-1}(\frac{1}{t})$ for $t > 0$. Differentiating with respect to $t$, and the symmetry, shows that $X \sim Cauchy(0,1)$.

## 8.2 Exercises

**Exercise - 8.2.1** - *A continuous random variable, $X$, having the density,*

$$f(x) = c\frac{e^{(x-a)/b}}{(1+e^{(x-a)/b})^2}; \qquad x \in \mathbb{R},$$

*is called the logistic random variable. It is denoted by $X \sim Logist(a, b)$, $a \in \mathbb{R}$, $b > 0$. Find the constant $c$. Then plot on the same graph paper the density of $Cauchy(0,1)$, the density of $N(0,1)$ and the density of $Logist(0,1)$. Which density has the highest tails? Which has the lowest tails.*





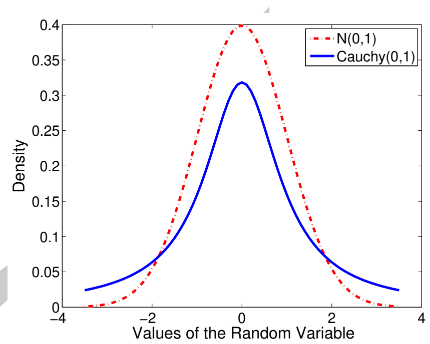



Figure 8.6: The Cauchy and Standard Normal Densities.

**Exercise - 8.2.2** - **(Chi square versus Poisson)** *Let $X$ have a chi square density with $n = 4$ degrees of freedom. Show that its cdf is:*

$$F(t) = 1 - \exp(-t/2)\sum_{j=0}^{2-1}\frac{(t/2)^j}{j!}, \qquad t > 0.$$

**Exercise - 8.2.3** - *Use a table of the distribution of $N(0,1)$, such as the table for $\Phi(t)$ provided in Example 8.1.5, to find $\mathbb{P}(-1.28 < X < 1.96)$.*

**Exercise - 8.2.4** - *Let $X \sim N(0,1)$. find $\mathbb{P}(X \le -2.5)$, $\mathbb{P}(X \ge 2.5)$, $\mathbb{P}(-2.5 < X < 2.5)$. You may use that table for $\Phi(t)$ provided in Example 8.1.5.*

**Exercise - 8.2.5** - *When $X$ is a standard normal random variable obtain the probability that $X$ could be in the interval $(0, 0.5)$ or $(2.5, 3.0)$. You may use that table for $\Phi(t)$ provided in Example 8.1.5.*

**Exercise - 8.2.6** - *When $X \sim Gamma(2,2)$, find the distribution $F(t) = \mathbb{P}(X \le t)$ for $t \in \mathbb{R}$.*

**Exercise - 8.2.7** - *Let $X \sim Beta(2,2)$. Find $\mathbb{P}(\frac{1}{2} < X < \frac{3}{4})$.*

**Exercise - 8.2.8** - *Let $X \sim \chi^2_{(4)}$. Find $\mathbb{P}(X < 4)$.*

**Exercise - 8.2.9** - *The human body temperature, $X$, is known to be normally distributed, i.e., $X \sim N(a, b^2)$, where $a = 98.6^oF$. If about 95% of the people have their typical body temperature between 98.2 and 99 degrees, what should be $b$? You may use that table for $\Phi(t)$ provided in Example 8.1.5.*

**Exercise - 8.2.10** - **(Radioactivity)** *For Plutonium 239 the half-life is 24,100 years. Model the time of decay of a Plutonium 239 atom with an exponential random variable. What is the rate of decay, $\lambda$? How long will it take for a Plutonium 239 dump site to decay to 60% of its original amount?*