CS-570 Computer Vision

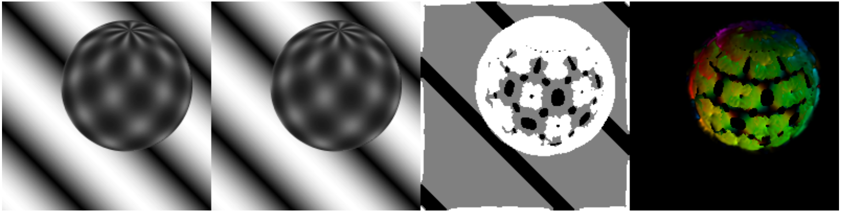
Nazar Khan

Department of Computer Science
University of the Punjab

17. Optic Flow – Global

Global Optic Flow

- In the last lecture, we covered the *local* optic flow method of Lucas & Kanade.
  - Simple and fast.
  - Low memory requirements.
  - Results often better than more sophisticated approaches.
  - Problems at locations where the local constancy assumption is violated: flow discontinuities and non-translatory motion (e.g. rotation).
  - Does not compute the flow field at all locations.



- In this lecture, we study a global method that produces dense flow fields (*i.e.*, at every pixel).

Variational Method of Horn & Schunck

- At some given time z the optic flow field is determined as minimising the function $(u(x,y), v(x,y))^T$ of the energy functional

$$E(U,V) = \frac{1}{2}\sum_{x,y}\left(\underbrace{(I_x u + I_y v + I_z)^2}_{\text{data term}} + \alpha \underbrace{\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)}_{\text{smoothness term}}\right)$$

- Has a unique solution that depends continuously on the image data.
- Global method since optic flow at (x,y) depends on all pixels in both frames.

Notation Alert!
U and V are 2D arrays of the same size as the frame. *Inside the summation* the flow component $u(x,y)$ at a pixel location is shortened as u. Similarly for v.

Variational Method of Horn & Schunck

$$E(U, V) = \frac{1}{2} \sum_{x,y} \left(\underbrace{(I_x u + I_y v + I_z)^2}_{\text{data term}} + \alpha \underbrace{\left(\|\nabla u\|^2 + \|\nabla v\|^2 \right)}_{\text{smoothness term}} \right)$$

- Regularisation parameter $\alpha > 0$ determines smoothness of the flow field.
  - $\alpha \to 0$ yields the normal flow.
  - The larger the value of α, the smoother the flow field.
- Dense flow fields due to filling-in effect:
  - At locations, where no reliable flow estimation is possible (small $\|\nabla I\|$), the smoothness term dominates over the data term.
- This propagates data from the neighbourhood.
- No additional threshold parameters necessary.

Functionals and Calculus of Variations

- Since U is a function, $E(U, V)$ is a function of a function. A function of a function is also called a *functional*.
- Standard calculus can optimize functions $f(x)$ by requiring $\frac{d}{dx} f|_{x^*} = 0$.
- Functionals are optimized via *calculus of variations*.
- Optimizer of an energy functional

$$E(U, V) = \sum_{x,y} F(x, y, u, v, u_x, u_y, v_x, v_y)$$

must satisfy the so-called *Euler-Lagrange* equations

$$\begin{aligned} \partial_x F_{u_x} + \partial_y F_{u_y} - F_u &= 0 \\ \partial_x F_{v_x} + \partial_y F_{v_y} - F_v &= 0 \end{aligned}$$

with some boundary conditions.

Functionals and Calculus of Variations

- For our energy functional $E(U, V)$,

$$F = \frac{1}{2}\left(I_x u + I_y v + I_z\right)^2 + \frac{\alpha}{2}\left(u_x^2 + u_y^2 + v_x^2 + v_y^2\right)$$

with partial derivatives

$$\begin{aligned}
F_u &= I_x\left(I_x u + I_y v + I_z\right)\\
F_v &= I_y\left(I_x u + I_y v + I_z\right)\\
F_{u_x} &= \alpha u_x\\
F_{u_y} &= \alpha u_y\\
F_{v_x} &= \alpha v_x\\
F_{v_y} &= \alpha v_y
\end{aligned}$$

Variational Method of Horn & Schunck

- So the Euler-Lagrange equations can be written as

$$\alpha(u_{xx} + u_{yy}) - I_x\left(I_x u + I_y v + I_z\right) = 0$$
$$\alpha(v_{xx} + v_{yy}) - I_y\left(I_x u + I_y v + I_z\right) = 0$$

- Laplacian $\Delta u = u_{xx} + u_{yy}$ can be written as $\frac{1}{h^2}\sum_{j\in\mathcal{N}_i}(u_j - u_i)$.
- At the ith pixel, after writing out the first and second order derivatives, we obtain

$$\frac{\alpha}{h^2}\sum_{j\in\mathcal{N}_i}(u_j - u_i) - I_{xi}\left(I_{xi} u_i + I_{yi} v_i + I_{zi}\right) = 0$$
$$\frac{\alpha}{h^2}\sum_{j\in\mathcal{N}_i}(v_j - v_i) - I_{yi}\left(I_{xi} u_i + I_{yi} v_i + I_{zi}\right) = 0$$

where h is the grid size (usually 1).

Variational Method of Horn & Schunck

- Two equations for every pixel.

$$\frac{\alpha}{h^2}\sum_{j\in\mathcal{N}_i}(u_j - u_i) - I_{xi}\left(I_{xi}u_i + I_{yi}v_i + I_{zi}\right) = 0$$
$$\frac{\alpha}{h^2}\sum_{j\in\mathcal{N}_i}(v_j - v_i) - I_{yi}\left(I_{xi}u_i + I_{yi}v_i + I_{zi}\right) = 0$$

- But computing (u_i, v_i) requires already knowing the flow (u_j, v_j) at the neighbours $j \in \mathcal{N}_i$.
- Approximate solution: start with an initial flow field $(U^{(0)}, V^{(0)})$.

Variational Method of Horn & Schunck

- Can be solved via *fixed-point iterations*

$$u_i^{(t+1)} = \frac{\frac{\alpha}{h^2}\sum_{j\in\mathcal{N}_i} u_j^{(t)} - I_{xi}\left(I_{yi}v_i^{(t)} + I_{zi}\right)}{\frac{\alpha}{h^2}|\mathcal{N}_i| + I_{xi}^2}$$

$$v_i^{(t+1)} = \frac{\frac{\alpha}{h^2}\sum_{j\in\mathcal{N}_i} v_j^{(t)} - I_{yi}\left(I_{xi}u_i^{(t)} + I_{zi}\right)}{\frac{\alpha}{h^2}|\mathcal{N}_i| + I_{yi}^2}$$

with $k = 0, 1, 2, \ldots$ and an arbitrary initialisation (e.g. zero vector).

Variational Method of Horn & Schunck

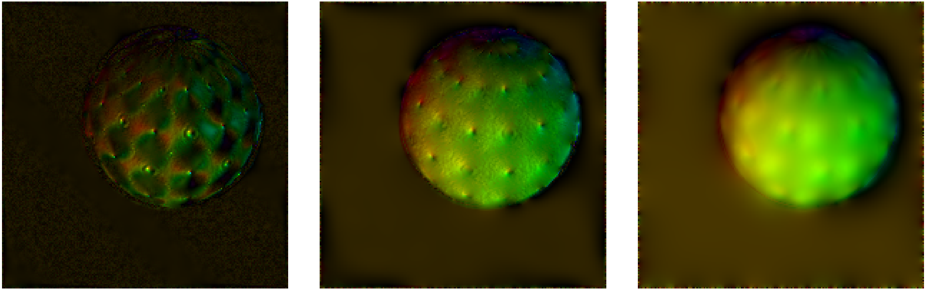


Figure: Left to right: Dense and smooth optic flow fields obtained via Horn & Schunck's variational method for smoothness parameter $\alpha = 0.0000001, 0.00001$ and 0.001 after 400 iterations. Noise smoothing scale was $\sigma = 0.5$. Author: N. Khan (2018)

Variational Method of Horn & Schunck

Variational Method of Horn & Schunck

Variational Method of Horn & Schunck

Figure: Left to right: Dense and smooth optic flow fields obtained via Horn & Schunck's variational method for smoothness parameter $\alpha = 0.0001$ after 400 iterations. Noise smoothing scale was $\sigma = 0.5$. Author: N. Khan (2018)

Variational Method of Horn & Schunck

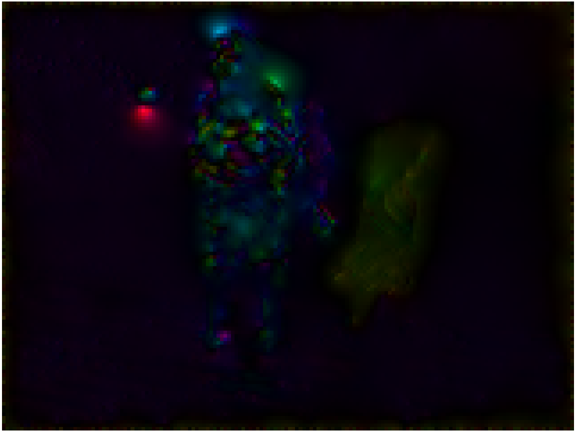


Figure: Left to right: Dense and smooth optic flow fields obtained via Horn & Schunck's variational method for smoothness parameter $\alpha = 0.001$ after 400 iterations. Noise smoothing scale was $\sigma = 0.5$. Author: N. Khan (2018)

Variational Method of Horn & Schunck

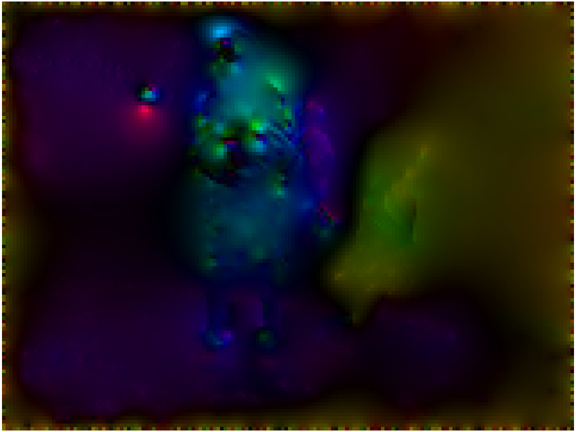


Figure: Left to right: Dense and smooth optic flow fields obtained via Horn & Schunck's variational method for smoothness parameter $\alpha = 0.01$ after 400 iterations. Noise smoothing scale was $\sigma = 0.5$. Author: N. Khan (2018)

Variational Method of Horn & Schunck
Summary

- Variational methods for computing optic flow are global methods.
- Create dense flow fields by filling-in.
- Model assumptions of the variational Horn & Schunck approach:
  1. grey value constancy,
  2. smoothness of the flow field
- Mathematically well-founded method.
- Minimising the energy functional leads to coupled differential equations.
- Variational methods can be extended and generalised in numerous ways, with respect to both models and algorithms.