MS-252 Linear Algebra

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5. Matrix Transformations

Matrix Transformations

- Special class of functions that arise from matrix multiplication.
- These functions are called "matrix transformations".
- Fundamental in the study of linear algebra.
- Important applications in physics, engineering, social sciences, and various branches of mathematics.

Basis Vectors

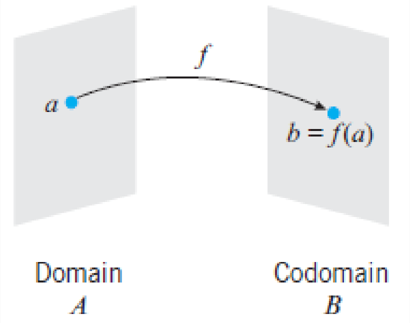
- Vectors. Default representation as a column of numbers. Denoted via lower-case, bold letters. For example, $\mathbf{x}, \mathbf{v}, \mathbf{b}$.
- Basis vectors for $\mathbb{R}^n$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

- Called basis vectors because *every* vector $\mathbf{x} \in \mathbb{R}^n$ can be represented as a linear combination of these basis vectors.

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

Functions



Usually we have considered functions as mappings from $a \in \mathbb{R}$ to $b \in \mathbb{R}$.

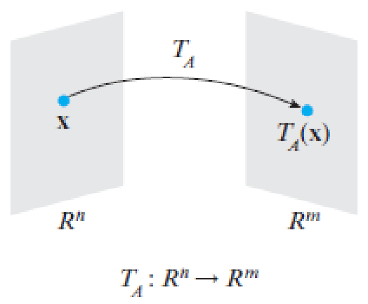
Transformations

- A function that maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ is usually called a *transformation*.
- Map vectors to vectors.
- Commonly denoted by the letter T.

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

- For $m = n$, the transformations are usually called *operators on $\mathbb{R}^n$*.
- Linear systems $A\mathbf{x}_{n\times 1} = \mathbf{b}_{m\times 1}$ can be viewed as transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Linear systems as Transformations



Represented as

- $T_A : \mathbb{R}^n \to \mathbb{R}^m$
- $\mathbf{b} = T_A(\mathbf{x})$
- $\mathbf{x} \xrightarrow{T_A} \mathbf{b}$.

Read as "T_A maps $\mathbf{x}$ onto $\mathbf{b}$".

Transformtion T_A is just "multiplication by matrix A".

Zero and Identity

- The $0_{m \times n}$ matrix containing all zeros is the *zero transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$T_0(\mathbf{u}) = \mathbf{0}$$

- I_n is the *identity operator* from $\mathbb{R}^n$ to $\mathbb{R}^n$.

$$T_{I_n}(\mathbf{u}) = \mathbf{u}$$

Properties

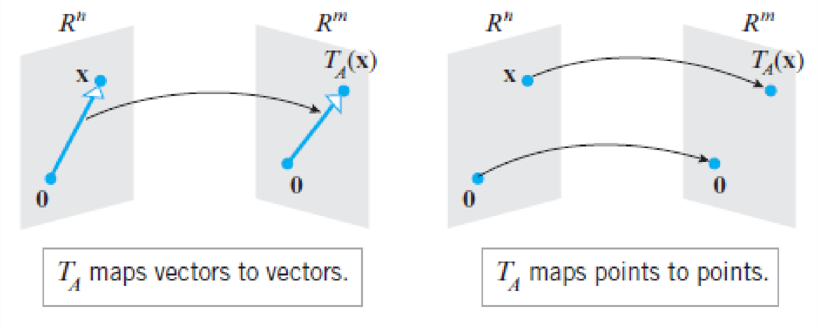
- For every matrix A the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:
  1. $T_A(\mathbf{0}) = \mathbf{0}$.
  2. $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$. *[Homogeneity property]*
  3. $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$. *[Additivity property]*
  4. $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$.
- Properties 2 and 3 imply *linearity*. A transformation with both properties is a *linear transformation*.
- Therefore, *matrix transformations are linear transformations*.
- It follows from 2 and 3 that

$$T_A(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_r\mathbf{u}_r) = k_1 T_A(\mathbf{u}_1) + k_2 T_A(\mathbf{u}_2) + \cdots + k_r T_A(\mathbf{u}_r)$$

  which states that *a matrix transformation maps a linear combination of vectors in $\mathbb{R}^n$ into the corresponding linear combination of vectors in $\mathbb{R}^m$*.

Vectors vs. Points

Since n-dimensional vectors can be viewed as points in $\mathbb{R}^n$, matrix transformations can be viewed as acting on vectors or points.



Which view to take depends on the problem.

Questions

- Exercise 1.8
  - 1, 3, 5, 7, 9, 11, 15, 21, 25, 27, 29, 31