MS-252 Linear Algebra

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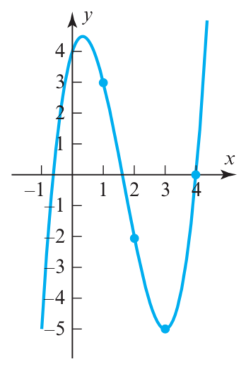
6. Polynomial Interpolation

Polynomial Interpolation

Given $n+1$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, we want to find a polynomial $P(x)$ of degree n such that:

$$P(x_i) = y_i \quad \text{for all } i = 0, 1, \ldots, n.$$



Problem Setup

Let $P(x)$ be a polynomial of degree n:

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

The interpolation conditions $P(x_i) = y_i$ lead to a system of linear equations:

$$\begin{cases} a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n = y_0, \\ a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n = y_1, \\ \vdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_n x_n^n = y_n. \end{cases}$$

Matrix Formulation

The system of equations can be written in matrix form as:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

The system matrix is also known as **Vandermonde matrix**.

Solving for $\mathbf{a} = [a_0, a_1, \dots, a_n]^T$ gives the coefficients of the interpolating polynomial.

Uniqueness of the Solution

Theorem: If the points $x_0, x_1, \ldots, x_n$ are distinct, the Vandermonde matrix is invertible, and the interpolating polynomial is unique.

Proof Sketch: The determinant of the Vandermonde matrix is non-zero when the x_i are distinct, ensuring invertibility.

Example

Example: Find the interpolating polynomial for the points $(1, 2)$, $(2, 3)$, and $(3, 5)$.

Solution: Set up the system:

$$\begin{cases} a_0 + a_1(1) + a_2(1)^2 = 2, \\ a_0 + a_1(2) + a_2(2)^2 = 3, \\ a_0 + a_1(3) + a_2(3)^2 = 5. \end{cases}$$

Solve the system to find a_0, a_1, a_2.

Approximate Integration

- There is no way to evaluate the integral

$$\int_0^1 \sin\left(\frac{\pi x^2}{2}\right) dx$$

  directly since there is no way to express an antiderivative of the integrand in terms of elementary functions.
- But we could find an interpolating polynomial and integrate the approximating polynomial.
- Consider the five points

$$x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75, \quad x_4 = 1$$

  that divide the interval $[0, 1]$ into four equally spaced subintervals.

Approximate Integration

- The values of

$$f(x) = \sin\left(\frac{\pi x^2}{2}\right)$$

at these points are approximately

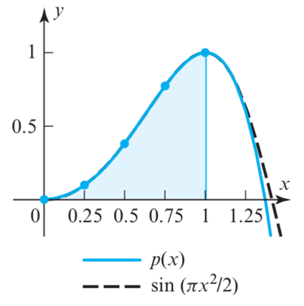
$$f(0) = 0, \quad f(0.25) = 0.098017, \quad f(0.5) = 0.382683,$$
$$f(0.75) = 0.77301, \quad f(1) = 1$$

- Verify that the interpolating polynomial is

$$p(x) = 0.098796x + 0.762356x^2 + 2.14429x^3 - 2.00544x^4$$

Approximate Integration

- The polynomial $p(x)$ can be used as an approximation of $f(x)$ over the interval $[0, 1]$.
- But more importantly, $p(x)$ can be integrated!



- Verify that

$$\int_0^1 p(x)dx \approx 0.438501$$