MA-310 Linear Algebra

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10. Inner Product Spaces

Inner Product

We used the dot product of vectors in $\mathbb{R}^n$ to define notions of

- length,
- angle,
- distance, and
- orthogonality.

Now we generalize those ideas to any vector space, not just $\mathbb{R}^n$.

Inner Product

An inner product on a real vector space V is a function that associates a real number $< \mathbf{u}, \mathbf{v} >$ with each pair of vectors in V in such a way that the following 4 axioms are satisfied for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.

1. $< \mathbf{u}, \mathbf{v} >=< \mathbf{v}, \mathbf{u} >$ **[Symmetry]**
2. $< \mathbf{u} + \mathbf{v}, w >=< \mathbf{u}, \mathbf{w} > + < \mathbf{v}, \mathbf{w} >$ **[Additivity]**
3. $< k\mathbf{u}, \mathbf{v} >= k < \mathbf{u}, \mathbf{v} >$ **[Homogeneity]**
4. $< \mathbf{v}, \mathbf{v} >\geq 0$ and $< \mathbf{v}, \mathbf{v} >= 0$ if and only if $\mathbf{v} = \mathbf{0}$ **[Positivity]**

A real vector space with an inner product is called a *real inner product space*.

Inner Product
Standard

- Inner product of two vectors **u** and **v** in $\mathbb{R}^n$ was earlier defined using the dot product

$$< \mathbf{u}, \mathbf{v} >= \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- This is commonly known as the *Euclidean inner product* or *standard inner product*.
- Inner product can be defined in other ways as well – as long as the defined function satisfies the 4 axioms in the last slide.

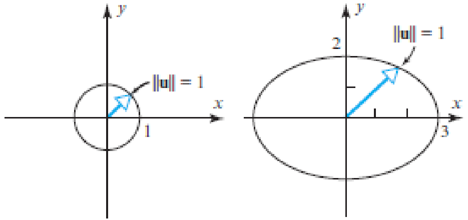
Weighted Euclidean Inner Product

- Defined as

$$< \mathbf{u}, \mathbf{v} >= w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

  with weights $w_1, w_2, \ldots, w_n$.
- Setting all weights to 1 yields the standard Euclidean inner product.

Weighted Euclidean Inner Product



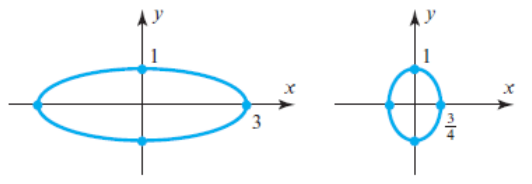
(a) The unit circle using the standard Euclidean inner product.

(b) The unit circle using a weighted Euclidean inner product.

- Left figure: Set of points at distance 1 from origin w.r.t standard Euclidean inner product $<\mathbf{u},\mathbf{v}>= u_1v_1 + u_2v_2$.
- Right figure: Set of points at distance 1 from origin w.r.t weighted Euclidean inner product $<\mathbf{u},\mathbf{v}>= \frac{1}{9}u_1v_1 + \frac{1}{4}u_2v_2$.

Weighted Euclidean Inner Product

- Sketch the unit circle in $\mathbb{R}^2$ w.r.t weighted Euclidean inner product $<\mathbf{u},\mathbf{v}>=\frac{1}{25}u_1v_1+\frac{1}{49}u_2v_2$.
- Find weighted Euclidean inner products on $\mathbb{R}^2$ for which the "unit circles" are the ellipses shown in the following figures.



Matrix inner product

- Defined as

$$< \mathbf{u}, \mathbf{v} >= A\mathbf{u} \cdot A\mathbf{v} = (A\mathbf{u})^T A\mathbf{v} = \mathbf{u}^T A^T A\mathbf{v}$$

- Also called the *inner product on $\mathbb{R}^n$ generated by A*.
- Setting $A = I$ yields the standard Euclidean inner product.
- Setting A as a diagonal matrix yields the weighted Euclidean inner product. Find A for $< \mathbf{u}, \mathbf{v} >= w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$.
- Can be viewed as standard inner product but after transforming by A.
- Plays a big role in Machine Learning, Image Processing, and Computer Vision.

Angles & Orthogonality
In General inner product spaces

- We have already seen that angle between two vectors in $\mathbb{R}^n$ can be computed using the dot product as

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}\right)$$

- Recall that dot product is a specialized form of inner product which is more general.
- *Angle between two vectors* in a *general inner product space* can be computed using the inner product as

$$\theta = \cos^{-1}\left(\frac{<\mathbf{u},\mathbf{v}>}{||\mathbf{u}||||\mathbf{v}||}\right)$$

Angles & Orthogonality
In General inner product spaces

- Recall that $-1 \leq \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} \leq 1$.
- For general inner products $-1 \leq \frac{<\mathbf{u},\mathbf{v}>}{||\mathbf{u}||||\mathbf{v}||} \leq 1$ also holds.
- *Norm (or length)* is defined by $||\mathbf{v}|| = \sqrt{<\mathbf{v},\mathbf{v}>}$.
- *Distance* between two vectors becomes $d(\mathbf{u},\mathbf{v}) = ||\mathbf{u}-\mathbf{v}|| = \sqrt{<\mathbf{u}-\mathbf{v},\mathbf{u}-\mathbf{v}>}$.
- Properties of length and distance also carry over in general spaces.
  - $||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ (*Triangle inequality for vectors*)
  - $d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$ (*Triangle inequality for distances*)
- $<\mathbf{u},\mathbf{v}>= 0$ implies *orthogonality*.
  - Note that orthogonality depends on the definition of the inner product.
  - Compute $<\mathbf{u},\mathbf{v}>$ for $\mathbf{u} = (1,1)$ and $\mathbf{v} = (1,-1)$ using standard and weighted Euclidean inner product definitions.

Example
Angle between square matrices

- We have seen that matrices satisfy the 10 axioms of vector spaces.
- For $n \times n$ matrices, an inner product can be defined as $< \mathbf{u}, \mathbf{v} >= \text{trace}(U^T V) = u_{11}v_{11} + u_{22}v_{22} + \cdots + u_{nn}v_{nn}$.
- Find the cosine of the angle between the vectors

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

- This gives us a method for computing similarities between objects in general vector spaces. *Prerequisite*: inner product needs to be defined first.