

# MA-310 Linear Algebra

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11. Gram-Schmidt Process

# Orthogonal & Orthonormal Sets

Content in this lecture applies to *sets* of 2 or more vectors.

- ▶ The solution of a problem can often be simplified by choosing a basis with orthogonal basis vectors.
- ▶ Further simplification is achieved if the orthogonal vectors are also unit vectors.

A *set* of two or more vectors in a real inner product space is said to be *orthogonal* if all pairs of distinct vectors in the set are orthogonal.

An orthogonal set in which each vector has norm 1 is said to be *orthonormal*.

- ▶ Roughly, orthonormal = orthogonal + normal.

# Orthogonal & Orthonormal Basis

- ▶ Recall that any set of  $n$  linearly independent vectors constitutes a basis for  $\mathbb{R}^n$ .
- ▶ If a set of basis vectors is orthogonal as well, it is called an *orthogonal basis*.
- ▶ If an orthogonal basis is made from unit vectors, it is called an *orthonormal basis*.
- ▶ Any vector  $\mathbf{v}$  can be normalized by dividing by its magnitude  $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$  which makes it a unit vector. [Verify this](#).
- ▶ A familiar orthonormal basis is the standard basis for  $\mathbb{R}^n$  with the Euclidean inner product:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

## Example

Verify that the following set is orthonormal with respect to the standard Euclidean inner product on  $\mathbb{R}^3$ .

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

## Why orthonormal basis?

- ▶ Assume  $S = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$  is a non-orthogonal basis for  $\mathbb{R}^n$ .
- ▶ Any vector  $\mathbf{u}$  can be represented in  $S$  as

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$$

which can be written as a linear system  $V\mathbf{c} = \mathbf{u}$  and solved for the coefficients  $c_1, c_2, \dots, c_n$ .

- ▶ Had  $S$  been orthogonal (or orthonormal), finding the  $c_i$  would have been much easier.
  - ▶ *Orthogonal case:*  $c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}$ .
  - ▶ *Orthonormal case:*  $c_i = \langle \mathbf{u}, \mathbf{v}_i \rangle$ .
- ▶ Notice the convenience – linear system versus simple inner products.

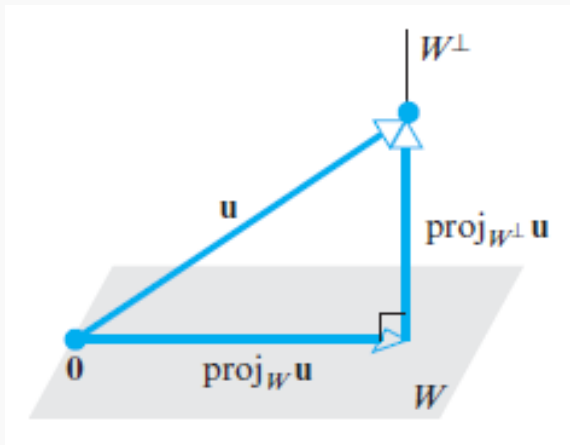
# Why orthonormal basis?

## *Proof of orthogonal case*

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}_i \rangle &= \langle c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n, \mathbf{v}_i \rangle \\ &= c_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \cdots + c_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle \\ &= c_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle \quad \text{since } S \text{ is an orthogonal basis} \\ \implies c_i &= \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2} \tag{1}\end{aligned}$$

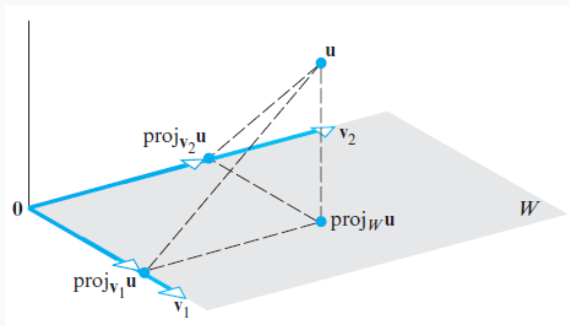
Orthonormal case can now be proven by observing that  $\|\mathbf{v}_i\| = 1$  if  $S$  is an orthonormal basis.

# Orthogonal Projection



- ▶  $\mathbf{u}$  is a vector in inner product space  $V$ .
- ▶  $W$  is a subspace of  $V$ .
- ▶ We can express

# Orthogonal Projections



- ▶ Let  $W$  be an  $r$ -dimensional subspace of  $V$  and let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$  be an orthogonal basis for  $W$ .
- ▶ 
$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2} \mathbf{v}_r.$$
- ▶ If basis is orthonormal, then
 
$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r.$$



# Gram-Schmidt Process

Every nonzero finite-dimensional inner product space has an orthonormal basis.

## The Gram-Schmidt Process

To convert a basis  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$  into an orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ , perform the following computations:

*Step 1.*  $\mathbf{v}_1 = \mathbf{u}_1$

*Step 2.*  $\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1$

*Step 3.*  $\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2$

*Step 4.*  $\mathbf{v}_4 = \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 - \frac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2} \mathbf{v}_3$

$\vdots$

(continue for  $r$  steps)

*Optional Step.* To convert the orthogonal basis into an orthonormal basis  $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_r\}$ , normalize the orthogonal basis vectors.