# MA-310 Linear Algebra

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PUCIT

6. Determinants

# Determinants

Content in this lecture applies only to square matrices.

- Gauss studied some quantities that *determine* some properties of a matrix.
- They are called *determinants*.
- For $2 \times 2$ matrices $det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
- This lecture is about determinants of general $n \times n$ matrices.

# Minors, Cofactors and a Recursive Formula for Determinants

For $n \times n$ matrix $A$,

- $M_{ij}$ =*minor of entry* $a_{ij}$=determinant of the submatrix that remains after the $i$th row and $j$th column are deleted from $A$.
- $C_{ij} = (-1)^{i+j} M_{ij}$ is called the *cofactor of entry* $a_{ij}$.
- For <u>any</u> row $i$

$$\mathrm{Det}(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

  which is recursive since $C_{ij}$ depends on the determinant of a smaller $(n-1) \times (n-1)$ matrix.

# Minors, Cofactors and a Recursive Formula for Determinants

- Also, for any column $j$

$$\mathrm{Det}(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

- Each cofactor $C_{ij}$ can in turn be computed in multiple ways.
- *Tip*: Pick row (or column) with maximium zeros. This will reduce computation.

> Whichever row or column is picked for cofactor expansions, the answer ($\det(A)$) will be the same.

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Historical note: An alternative method for computing determinants was invented by the author of *Alice's Adventures in Wonderland*. He was actually a mathematician.

# Practice

Find determinants of the following matrices

$$\begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

Be smart in picking the row or column for cofactor expansion.

# When is Det$(A) = 0$?

1. If $A$ has a row of zeros or a column of zeros, then $\det(A) = 0$.
   - Since cofactor expansion of all rows gives the same answer, let us pick the row of all zeros.
   - Let $i$ be the index of the row of zeros.
   - Then $\det(A) = 0C_{i1} + 0C_{i2} + \cdots + 0C_{in} = 0$.
   - Similarly for column of zeros.
2. If $A$ has two proportional rows or two proportional columns then $\det(A) = 0$.
   - Proof to follow.

# Determinant of diagonal and triangular matrices

- Determinant of lower triangular matrix can be computed as

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} \begin{vmatrix} a_{44} \end{vmatrix} = a_{11}a_{22}a_{33}a_{44}$$

- Same can be shown for upper triangular and diagonal matrices.

> Determinant of diagonal and triangular matrices is equal to product of diagonal entries.

# Determinants and EROs

| ERO | Effect on Determinant |
|---|---|
| Scale by $k$ | Scaled by $k$ |
| Add multiple of a row to another | No change |
| Swap two rows | Multiplied by $-1$. |

| Relationship | Operation |
|---|---|
| $\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ <br> $\det(B) = k\det(A)$ | In the matrix $B$ the first row of $A$ was multiplied by $k$. |
| $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ <br> $\det(B) = -\det(A)$ | In the matrix $B$ the first and second rows of $A$ were interchanged. |
| $\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ <br> $\det(B) = \det(A)$ | In the matrix $B$ a multiple of the second row of $A$ was added to the first row. |

# Determinants via EROs

- This gives us an alternative method for computing determinants.
  1. Reduce to triangular form via EROs.
  2. Take product of diagonal entries and the factors introduced because of the EROs.

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

The first and second rows of $A$ were interchanged.

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

A common factor of 3 from the first row was taken through the determinant sign.

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

$-2$ times the first row was added to the third row.

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

$-10$ times the second row was added to the third row.

$$= (-3)(-55) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

A common factor of $-55$ from the last row was taken through the determinant sign.

$$= (-3)(-55)(1) = 165$$

# Proportional rows/columns $\implies$ det$= 0$

*Proof*

- Row-echelon form is always upper-triangular.
- If matrix has two proportional rows/columns, row-echelon form will contain a row/column of zeros.
- So diagonal of row-echelon form will contain a 0.
- So determinant of row-echelon form will be 0.
- Since EROs can only scale the determinant, this means that determinant of original matrix must be 0 as well.

# Properties

- $\text{Det}(kA) = k^n\text{Det}(A)$.
- $\text{Det}(A + B) \neq \text{Det}(A) + \text{Det}(B)$.
- $\text{Det}(EB) = \text{Det}(E)\text{Det}(B)$. (See 4 slides back.)
- $\text{Det}(E_1 E_2 \dots E_r B) = \text{Det}(E_1)\text{Det}(E_2)\dots\text{Det}(E_r)\text{Det}(B)$.

# Determinant and Invertibility

$\boxed{A \text{ is invertible if and only if } \det(A) \neq 0.}$

Proof: Let $R$ be the RREF of $A$. Then $R = E_1 E_2 \dots E_r A$ and so

$$\text{Det}(R) = \text{Det}(E_1)\text{Det}(E_2)\dots\text{Det}(E_r)\text{Det}(A) \tag{1}$$

$A$ invertible $\implies R = I \implies \det(A) \neq 0$ since L.H.S of (1) $\neq 0$ and $\det(E_i) \neq 0$ always.

Similarly, $\det(A) \neq 0 \implies \det(R) \neq 0 \implies R$ does not have any zero row $\implies R = I$.

# Equivalent Statements

- If $A$ is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.
  1. $A$ is invertible.
  2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  3. The reduced row echelon form of $A$ is $I_n$.
  4. $A$ is expressible as a product of elementary matrices.
  5. $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ vector $\mathbf{b}$. The solution is $\mathbf{x} = A^{-1}\mathbf{b}$.
  6. $\det(A) \neq 0$.

# Det$(AB)$

$\boxed{\text{Det}(AB) = \text{Det}(A)\text{Det}(B).}$

Proof: $A$ is either invertible or not invertible.

$$\begin{aligned} A \text{ invertible} \quad &\implies A = E_1 E_2 \dots E_r \\ &\implies AB = E_1 E_2 \dots E_r B \\ &\implies \det(AB) = \det(E_1 E_2 \dots E_r B) \\ &\qquad\qquad\quad = \det(E_1)\det(E_2)\dots\det(E_r)\det(B) \\ &\qquad\qquad\quad = \det(E_1 E_2 \dots E_r)\det(B) \\ &\qquad\qquad\quad = \det(A)\det(B) \end{aligned}$$

$$\begin{aligned} A \text{ not invertible} \quad &\implies \det(A) = 0 \implies \det(A)\det(B) = 0 \\ A \text{ not invertible} \quad &\implies AB \text{ not invertible} \\ &\implies \det(AB) = 0 \end{aligned}$$

*So Det$(AB)$ =Det$(A)$Det$(B)$ always.*

# $\det(A^{-1})$

$$\boxed{\text{For invertible } A,\ \det(A^{-1}) = \frac{1}{\det(A)}}$$

Proof: $A$ invertible $\implies AA^{-1} = I \implies \det(AA^{-1}) = 1 \implies \det(A)\det(A^{-1}) = 1 \implies \det(A^{-1}) = \frac{1}{\det(A)}$ since $\det(A) \neq 0$.

# Adjoint

- Let $C_{ij}$ be the cofactor of entry $a_{ij}$ of $n \times n$ matrix $A$.
- Then the *adjoint matrix* is defined as

$$\mathrm{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}^T$$

- Notice the transpose.
- Show that adjoint of $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$ is $\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$

# A Formula for Matrix Inverse

- Recall that for any row $i$

$$\text{Det}(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

- If entries come from row $i$ and cofactors come from row $j \neq i$, then *the answer is always zero*. Verify.
- Consider the product $A\text{adj}(A)$.

$$A\,\text{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{j1} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{j2} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{jn} & \dots & C_{nn} \end{bmatrix}$$

# A Formula for Matrix Inverse

- The blue highlighted row and column product is
  - 0 for $i \neq j$, and
  - $\det(A)$ for $i = j$.
- So

$$A\text{adj}(A) = \begin{bmatrix} \det(A) & 0 & \dots & 0 \\ 0 & \det(A) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \det(A) \end{bmatrix} = \det(A)I$$

- Therefore, $A\left(\frac{1}{\det(A)}\text{adj}(A)\right) = I$.
- This gives us a *formula* for matrix inversion.

> If $A$ is invertible, then $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$.

# Cramer's Rule

- If $A\mathbf{x} = \mathbf{b}$ is a system of $n$ linear equations in $n$ unknowns such that $\det(A) \neq 0$, then the system has a unique solution given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)},$$

  where matrix $A_j$ is obtained by replacing the $j$th column of $A$ by $\mathbf{b}$.
- Proof:
- Advantages
  - No matrix inverse. Only determinants.
  - Solve for one variable at a time.
  - Easier for humans.
- Disadvantage
  - Solve for one variable at a time.
  - Slow for a computer.