# MA-310 Linear Algebra

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8. General Vector Spaces

# General Vector Spaces

If a set of objects satisfies some basic properties of vectors in $\mathbb{R}^n$, then those objects can be treated as vectors too.

*Axiom*: An assumption that is taken to be true without proof. They serve as a starting point.

Time to unlearn what we have been taught!

# General Vector Spaces

Any object can be treated as a vector.

Operator '$+$' can be redefined according to our needs.

Operator '$\times$' can be redefined according to our needs.

# Addition of objects

- Let $V$ be a set of objects and $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be members of this set.
- *Addition* is defined as an *operator* on objects in $V$.
- Denoted by the symbol '$+$'.
- Result $\mathbf{u} + \mathbf{v}$ of addition is called the *sum*.

# Scalar multiplication of objects

- Let $k$ be any scalar.
- *Scalar multiplication* is defined as an *operator* on objects in $V$.
- Denoted by the symbol '$\times$'.
- Result $k\mathbf{u}$ of multiplication is called the *product*.

So far in your life, $V$ has been the set of real numbers. But what stops it from being a set of other (any) kinds of objects!

# General Vector Spaces

- Notice that for real vector spaces, $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ were still members of $V$.
- If the objects in a general set $V$ also satisfy these properties, then they also form a vector space.
- Specifically, to qualify as a vector space, objects in $V$ must satisfy

| | | |
|---|---|---|
| 1. | $\mathbf{u}+\mathbf{v} \in V$ | Closure under addition |
| 2. | $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ | |
| 3. | $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$ | |
| 4. | $\mathbf{u}+\mathbf{0} = \mathbf{u}$ and $\mathbf{0} \in V$ | Zero vector |
| 5. | $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$ for every $\mathbf{u}$ and $-\mathbf{u} \in V$ | Negative |
| 6. | $k\mathbf{u} \in V$ | Closure under scalar multiplication |
| 7. | $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$ | |
| 8. | $(k+m)\mathbf{u} = k\mathbf{u}+m\mathbf{u}$ | |
| 9. | $k(m\mathbf{u}) = (km)\mathbf{u}$ | |
| 10. | $1\mathbf{u} = \mathbf{u}$ | |

# General Vector Spaces

- $\mathbf{u}$ could be an $n$-tuple, a 2-D array (matrix), an $N$-D array (tensor), an image, a video, a document, an X-ray, a brain-scan, an email, ...
- **As long as the objects satisfy the 10 vector space axioms, they can be treated as vectors in a general vector space.**

# Examples of sets that are vector spaces

- The zero vector space.
- $\mathbb{R}^n$.
- $\mathbb{R}^\infty$.
- $\mathbb{R}^{m\times n}$ – the set of all $m \times n$ matrices.
- The vector space of real-valued functions.

# Examples of sets that are *not* vector spaces

- $\mathbb{R}^{n+}$ – the set of $n$-tuples of positive real numbers. Why?
- $V = \mathbb{R}^2$ with scalar multiplication defined as $k\mathbf{u} = (ku_1, 0)$. Why?

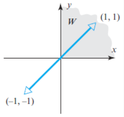
# Subspaces

A subset $W$ of vector space $V$ is called a *subspace* of $V$ if $W$ is itself a vector space.

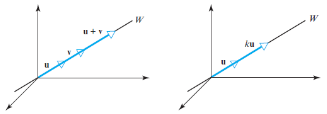
- Any subset of a vector space will automatically satisfy axioms $2, 3, 7, 8, 9$ and $10$.
- If it satisfies 1 and 6 (additive and multiplicative closures), then it will also satisfy 4 and 5. Why?
  - For $\mathbf{u} \in W$, axiom 6 implies $k\mathbf{u} \in W$.
  - Setting $k = 0$ and $k = -1$ implies $\mathbf{0} \in W$ and $-\mathbf{u} \in W$.
  - Finally axiom 1 then implies axioms 4 and 5 are true.
- Therefore, to verify if a subset $W$ of vector space $V$ is a subspace of $V$, one only needs to verify if objects in $W$ satisfy axioms 1 and 6 (*i.e.* is $W$ closed under addition and scalar multiplication?).

# Subspaces
*Examples*

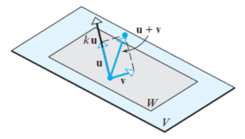
- $\mathbb{R}^{2++}$ is a subset but not a subspace of $\mathbb{R}^2$.
- Any line through the origin is a subspace of $\mathbb{R}^2$. All other lines are just subsets since they do not contain a $\mathbf{0}$ vector.
- Any line or plane through the origin is a subspace of $\mathbb{R}^3$. All other lines and planes are just subsets.
- Symmetric matrices constitute a subspace of the vector space of all square matrices.




$W$ is a not a subspace of $\mathbb{R}^2$.




$W$ is a subspace of $\mathbb{R}^3$.



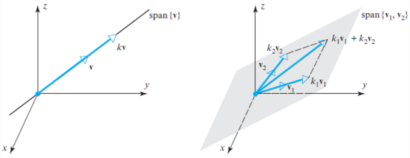

$W$ is not subspace of $V$.

# Span

- *Span* of a set of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ is the set of all vectors that can be generated from their *linear combinations*.

$$\text{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r) = k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_r\mathbf{u}_r$$

  where the *coefficients* $k_i$ are scalars between $-\infty$ and $\infty$.
- Span of $\mathbf{u}$ is $k\mathbf{u}$ which is a line in the direction of $\mathbf{u}$.
- Span of $\mathbf{u}$ an $\mathbf{v}$ is a plane containing both vectors.
- Span of standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is $\mathbb{R}^n$.




(*a*) Span {**v**} is the line through the origin determined by **v**.

(*b*) Span {$\mathbf{v}_1, \mathbf{v}_2$} is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

# Testing for Linear Combination

Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in $\mathbb{R}^3$. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and that $\mathbf{w}' = (4, -1, 8)$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

# Testing for spanning

Determine whether the vectors $\mathbf{v}_1 = (1, 1, 2), \mathbf{v}_2 = (1, 0, 1), and \mathbf{v}_3 = (2, 1, 3)$ span the vector space $\mathbb{R}^3$.

If $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ span $\mathbb{R}^3$, then $\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$ should be true for *all* $\mathbf{b} \in \mathbb{R}^3$. This can be written as

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

This linear system has a solution for all $\mathbf{b}$ if and only if the system matrix is invertible. This one is not. So $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ do not span $\mathbb{R}^3$.

# Linear Independence

Definition

Set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ of two or more vectors in a vector space $V$, is a *linearly independent set* if no vector in $S$ can be expressed as a linear combination of the others. A set that is not linearly independent is said to be linearly dependent.

Test for linear independence

$S$ is linearly independent if and only if the only coefficients satisfying the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

are $k_1 = 0, k_2 = 0, \ldots, k_r = 0$.

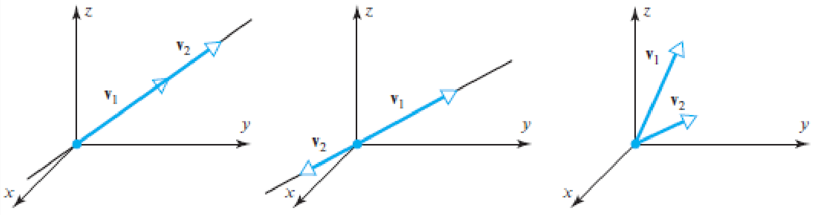
Proof by contradiction.

# Linear Independence

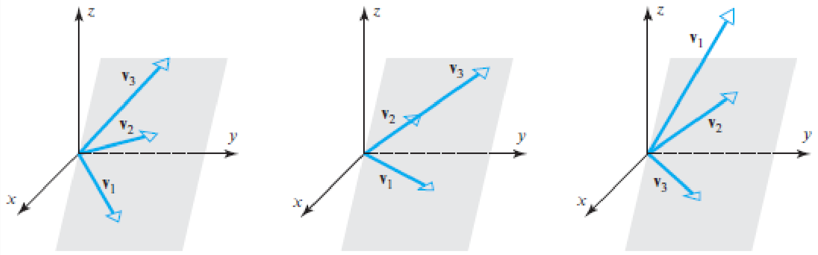
Determine whether the vectors $\mathbf{v}_1 = (1, -2, 3), \mathbf{v}_2 = (5, 6, -1), \mathbf{v}_3 = (3, 2, 1)$ are linearly independent or not.

# Linear Independence

## *Geometric Interpretation*




(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent




(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent

# Linear Independence

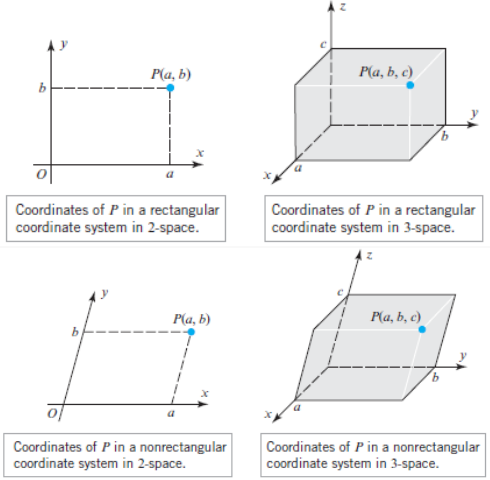
> Let $S = \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be a set of $r$ vectors in $\mathbb{R}^n$. If $r > n$, then $S$ *must be* linearly dependent.

Proof:

The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$ corresponds to a homogenous linear system with $n$ equations and $r$ unknowns. For $r > n$, it will have non-trivial solutions and hence the set $S$ will be linearly dependent.

# Coordinate Systems

- We usually work in *rectangular coordinate systems*.
- They are convenient but not necessary.



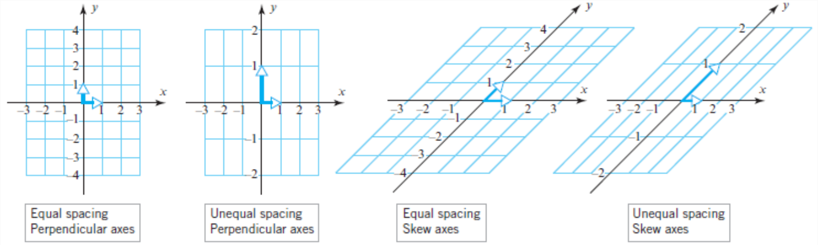

Coordinates of $P$ in a rectangular coordinate system in 2-space.

Coordinates of $P$ in a rectangular coordinate system in 3-space.

Coordinates of $P$ in a nonrectangular coordinate system in 2-space.

Coordinates of $P$ in a nonrectangular coordinate system in 3-space.

# Non-rectangular, unequal coordinate systems



# Basis

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in a finite-dimensional vector space $V$ , then $S$ is called a basis for $V$ if:

1. $S$ spans $V$.
2. $S$ is linearly independent.

Examples:

- Standard basis for $\mathbb{R}^n$.
- Any set of $n$ linearly independent vectors in $\mathbb{R}^n$. (Show that the vectors $\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$.)
- Standard basis for $M_{mn}$.

# Benefit of Basis

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space $V$, then every vector $\mathbf{v}$ in $V$ can be expressed in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in *exactly one way*.

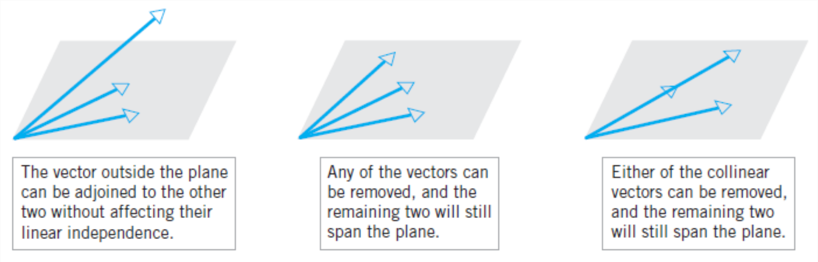
Proof: $S$ is a basis $\implies$ $\mathbf{v}$ can be expressed in *some* way.
Assume $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ and also $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$.
Subtracting both leads to $\mathbf{0} = (c_1 - k_1)\mathbf{v}_1 + (c_2 - k_2)\mathbf{v}_2 + \cdots + (c_n - k_n)\mathbf{v}_n$.
Linear independence of $S$ $\implies$ $(c_i - k_i) = 0$. Therefore, there can be exactly one representation of $\mathbf{v}$ in a basis.

Scalers $c_1, c_2, \ldots, c_n$ are called *coordinates* of $\mathbf{v}$ relative to basis $S$. Vector $(c_1, c_2, \ldots, c_n)$ is called the *coordinate vector* of $\mathbf{v}$ relative to basis $S$.

# Dimension

- The number of vectors in a basis for $V$ is called the *dimension* of $V$.
- Denoted as $\dim(V)$.
- All basis of $V$ must have the same dimension. Why?
- Zero vector space has dimension 0. That is $\dim(\{\mathbf{0}\}) = 0$.
- In engineering as well as computer science, dimension is sometimes referred to as *degrees of freedom*.

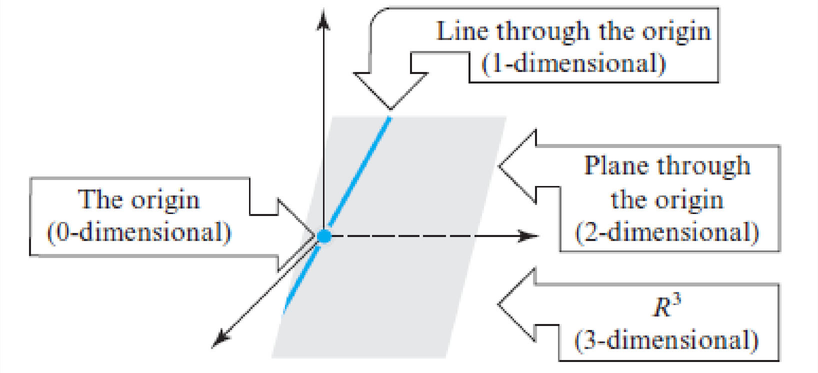
# Plus/Minus Theorem




Consequences:

- If $V$ has dimension $n$, then for any subset $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, it suffices to check *either* linear independence *or* spanning – the remaining condition will hold automatically.
- If $S$ spans $V$ but is not a basis for $V$, then $S$ can be reduced to a basis for $V$ by removing appropriate vectors from $S$.
- If $S$ is a linearly independent set that is not already a basis for $V$.

# Dimension
## *Geometric view*



# Change of Basis

- A basis that is suitable for one problem may not be suitable for another.
- So it is common to change from one basis to another.