MA-310 Linear Algebra

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9. Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors
Definition

Content in this lecture applies only to square matrices.

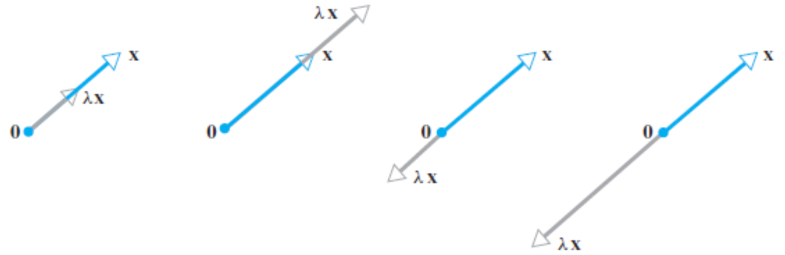
- Recall that matrix-vector multiplication $\implies$ linear transformation.
- So every matrix-vector multiplication $M\mathbf{v}$ transforms vector $\mathbf{v}$.
- This transformation includes direction as well as scale.
- However, for a given M there are some nonzero vectors that are only scaled. That is

$$M\mathbf{v} = \lambda\mathbf{v} \tag{1}$$

  where λ is the scaling factor.
- Such vectors are called *eigenvectors* of M and the corresponding scales λ are called *eigenvalues*.

Eigenvalues and Eigenvectors
Definition

Cases when $M\mathbf{x}$ (in gray) is only a scaled version of $\mathbf{x}$ (in blue).



(a) $0 \leq \lambda \leq 1$ (b) $\lambda \geq 1$ (c) $-1 \leq \lambda \leq 0$ (d) $\lambda \leq -1$

Eigenvalues and Eigenvectors
History

- Derived from the German word *eigen*, meaning "own", "peculiar to", "characteristic", or "individual".
- Every square matrix has its own particular vectors that do not change direction after multiplication.

Eigenvalues and Eigenvectors
Uses

Applications in such diverse fields as

- computer graphics
- mechanical vibrations
- heat flow
- population dynamics
- quantum mechanics
- economics
- machine learning
- computer vision
- Google's PageRank algorithm
- lots of other areas.

Eigenvalues and Eigenvectors
How to compute?

- If $\mathbf{v}$ is an eigenvector of M with corresponding eigenvalue λ, then

$$M\mathbf{v} = \lambda\mathbf{v} \implies \lambda\mathbf{v} - M\mathbf{v} = \mathbf{0} \implies (\lambda I - M)\mathbf{v} = \mathbf{0}$$

  which implies that $\mathbf{v}$ is a null-vector of $\lambda I - M$.
- Since $\mathbf{v}$ is constrained to be nonzero, $\lambda I - M$ must have a null space (*i.e.*, 0 determinant)

$$\det(\lambda I - M) = 0$$

  which is called the *characteristic equation of M*. This equation is used to find eigenvalues and eigenvectors.
- Compute characteristic equation for $M = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$.

Eigenvalues and Eigenvectors

How to compute?

- When the determinant is expanded, the characteristic equation of M takes the form

$$\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n$$

which is called the *characteristic polynomial of M*.

- Since it is always of degree n, it can have *maximum* n distinct roots.
- Therefore, an $n \times n$ matrix can have a maximum of n *distinct* eigenvalues.
- An eigenvalue can sometimes be a complex number.

Examples

For $M = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$,

characteristic equation is $\begin{vmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{vmatrix} = 0$

$\implies (\lambda+1)\lambda - (-3)(-2) = 0$

$\implies \lambda^2 + \lambda - 6 = 0$ (L.H.S is called the *characteristic polynomial*)

$\implies \lambda = 2$ and $\lambda = -3$ are the 2 eigenvalues of M.

For eigval $\lambda = -3$, $(\lambda I - M)\mathbf{v} = \mathbf{0}$

$$\implies \begin{bmatrix} -3+1 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_2 = -\frac{2}{3} v_1.$$

So the basis for the eigenspace corresponding to $\lambda = -3$ is the vector $\mathbf{v} = \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix}$

Examples

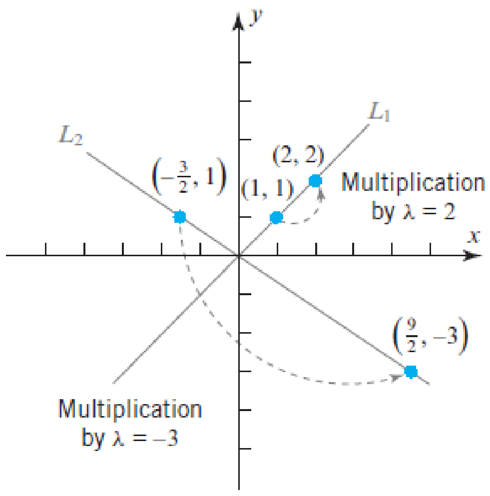
For eigval $\lambda = 2$, $(\lambda I - M)\mathbf{v} = \mathbf{0}$

$$\implies \begin{bmatrix} 2+1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_2 = v_1.$$

So the basis for the eigenspace corresponding to $\lambda = 2$ is the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Examples


y
x
L_1
L_2
(2, 2)
(1, 1)
$\left(-\frac{3}{2}, 1\right)$
$\left(\frac{9}{2}, -3\right)$
Multiplication by λ = 2
Multiplication by λ = –3

Examples

Geometry of eigenvectors of the matrix $M = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$.

- The eigenspace corresponding to $\lambda = 2$ is the line L_1 through the origin and the point $(1, 1)$.
- The eigenspace corresponding to $\lambda = 3$ is the line L_2 through the origin and the point $(-\frac{3}{2}, 1)$.
- Multiplication by M maps each vector in L_1 back into L_1, scaling it by a factor of 2.
- Similarly, each vector in L_2 is mapped back into L_2 after scaling it by a factor of -3.

Examples

- For $M = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$, characteristic polynomial is $(\lambda - 1)(\lambda - 2)^2 = 0$. Verify.
- 2 is a root of the polynomial with multiplicity 2.
- Their will be 2 eigenvectors corresponding to eigenvalue 2.
- We can also say that the eigenspace corresponding to $\lambda = 2$ will be 2-dimensional. Find it.

Triangular Matrices

- Eigenvalues of any triangular matrix (lower, upper or diagonal) are the entries on the main diagonal.
- Proof: Look at the characteristic polynomial of $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ or any other diagonal matrix. Complete the proof.

Similarity Transformation

- Take two $n \times n$ matrices A and P.
- Assume P to be invertible.
- Consider the transformation

$$A \to P^{-1}AP$$

  It is called a *similarity transformation*.
- If $B = P^{-1}AP$, then A and B are said to be *similar matrices*.

Similarity Transformation

- Such transformations are important because they preserve many properties of A. A and $P^{-1}AP$ have the same
  - Determinant
  - Invertibility
  - Rank
  - Nullity
  - Trace
  - Characteristic polynomial
  - Eigenvalues
  - Eigenspace dimension

Diagonalization
Definition

- We have seen that diagonal matrices are very convenient.
  - Easily invertible.
  - Eigenvalues are the diagonal entries themselves.
  - Powers are easy.

> For $n \times n$ matrices A and P where P is invertible, if $P^{-1}AP$ turns out to be a diagonal matrix, then A is said to *diagonalizable* and P is said to *diagonalize* A.

- If A is similar to a diagonal matrix, then many properties of A can be obtained through the more convenient diagonal matrix $P^{-1}AP$.

Diagonalization
Method

- <u>Assume</u> A is similar to a diagonal matrix D.
- <u>Then</u> P exists such that $P^{-1}AP = D \implies AP = PD \implies$ $A[\, \mathbf{p}_1 \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_n \,] = [\, \mathbf{p}_1 \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_n \,]\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \implies A\mathbf{p}_i = d_i\mathbf{p}_i$ for $i = 1, 2, \ldots, n$.
- Therefore, the matrix P that diagonalizes A is made from the n eigenvectors of A.
- Since P is invertible (by assumption), the n eigenvectors must be linearly independent.
- Also, the diagonal matrix D is made from the corresponding eigenvalues of A.

Diagonalization
Method

- Now assume A has n linearly independent eigenvectors.
- This implies that P made from those eigenvectors is invertible.
- This implies that A is diagonalizable.
- So we can state the following.

> If A is an $n \times n$ matrix, the following statements are equivalent.
>
> 1. A is diagonalizable.
> 2. A has n linearly independent eigenvectors.

Eigenvalues of A^k

$$A^k\mathbf{x} = A^{k-1}A\mathbf{x} = A^{k-1}\lambda\mathbf{x} = \lambda A^{k-1}\mathbf{x} = \lambda A^{k-2}A\mathbf{x} = \lambda A^{k-2}\lambda\mathbf{x} = \lambda^2 A^{k-2}\mathbf{x} = \cdots = \lambda^k\mathbf{x}.$$

If λ is an eigenvlaue of A with corresponding eigenvector $\mathbf{x}$, then λ^k will be an eigenvlaue of A^k with the same corresponding eigenvector $\mathbf{x}$.

Computing A^k via diagonalization

$$
\begin{aligned}
& P^{-1}AP = D \\
& \implies (P^{-1}AP)^2 = D^2 \\
& \implies (P^{-1}AP)(P^{-1}AP) = D^2 \\
& \implies P^{-1}APP^{-1}AP = D^2 \\
& \implies P^{-1}AIAP = D^2 \\
& \implies P^{-1}A^2P = D^2 \\
& \implies A^2 = PD^2P^{-1}
\end{aligned}
$$

More generally, for any positive integer k, $A^k = PD^kP^{-1}$.
Notice that computing D^k is much easier.

Example

We have already computed the eigen-decomposition for $M = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Use it to compute A^{13}.

Geometric and Algebraic Multiplicity

- *Geometric multiplicity:* Dimension of the eigenspace corresponding to an eigenvalue.
- *Algebraic multiplicity:* Number of times an eigenvalue appears as a solution of the characteristic polynomial.
- Algebraic multiplicity $\geq$ geometric multiplicity.

> A square matrix is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.