MA-250 Probability and Statistics

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Lecture 11

Terminology

Random Experiment

- Anything that produces an uncertain output.
- Tossing a coin, rolling a die, voting in elections, etc.

Outcome

- What an experiment produces.
- Coin landing heads, die giving a 6, election resulting in People's Party winning.

Terminology

Events

- The class E of all events that we are interested in is also called a sigma field. It obeys the following axioms
  1. S is always considered an event,
  2. If A is an event then A^c must also be considered as an event,
  3. A countable union of events must also be an event. That is, if $A_1, A_2, \cdots$ are all events then $A_1 \cup A_2 \cup \cdots$ must also be an event.

Terminology

When we perform a random experiment.

- S = sample space
- E = events in the sample space
- P = real-valued probability function for events E ($P(E) \rightarrow [0,1]$)

<u>Probability Space</u>: The collection (S,E,P).

Axioms of Probability

The probability function P obeys the following axioms:

1. $0 \leq P(A) \leq 1$ for any event A,
2. $P(S) = 1$ and
3. If $A_1, A_2, \ldots$ are mutually exclusive events then $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

Properties of P

If A,B are events, then

1. $P(\emptyset) = 0$, (impossibility property)
2. $P(A^c) = 1 - P(A)$, (complement property)
3. $P(A^c \cap B) = P(B) - P(A \cap B)$, (more general complement property)
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, (union property)
5. If $A \subseteq B$, then $P(A) \leq P(B)$, (monotonicity property)

METHODS FOR COMPUTING PROBABILITIES

Methods for Computing Probabilities

1. By Counting Elements
2. By Measuring Sizes

Probabilities via Counting Elements

Simple Sample Space

A finite sample space $S=\{\omega_1,\omega_2,\ldots,\omega_n\}$ in which every outcome ω_i is equally likely

- $P(\{\omega_i\})=1/n$
- $p_1+p_2+\ldots+p_n=1$

P(A) = (# elements in A) / (# elements in S)

= |A| / |S|

Probabilities via Counting Elements

- **For a simple sample space in which the elements can be counted, probabilities can be computed via counting.**

Probabilities via Counting Elements

- **Experiment**: Tossing 4 coins
- S = ? |S|=16
- **If S is a simple sample space**, then every outcome is equally likely: $P(\omega_i)=1/16$
- A = getting 3 heads = {HHHT, HHTH, HTHH, THHH}
- P(HHHT ∪ HHTH ∪ HTHH ∪ THHH)

  = P(HHHT) + P(HHTH) + P(HTHH) + P(THHH)

  = 1/16 + 1/16 + 1/16 + 1/16

  = 4/16 = |A| / |S|

Probabilities via Counting Elements

If S is <u>not</u> a simple sample space, then every outcome is <u>not</u> equally likely

- $P(\omega_i) \neq P(\omega_j)$
- $P(A) \neq |A| / |S|$

- **Experiment**: Roll a fair die twice and note the sum of outcomes.
- Define the sample space?
- S_1={2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
- Let P_1 be the probability measure for events in S_1.
- What is $P_1(\{3\})$?

Consider another sample space

$$S_2=\left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}\right\}$$

Entry (i,j) in S_2 corresponds to the event

{i on die 1, j on die 2}.

- When will S_2 be a simple sample space?
  - Assuming fair dice, all combinations {i,j} are equally likely and S_2 is a simple sample space.
- So we can compute

$$P_1(\{3\}) = P_2(\{1,2\}) + P_2(\{2,1\}) = 1/36 + 1/36 = 2/36 = 1/18$$

- Probability space (S_2,P_2) can be used to answer all questions about the experiment.
- Can we use probability space (S_1,P_1) to find the probability of the event A={both faces are even}?

Secretary's Matching Problem

I want to send letters to N different people.

Letters	Envelopes
N	N

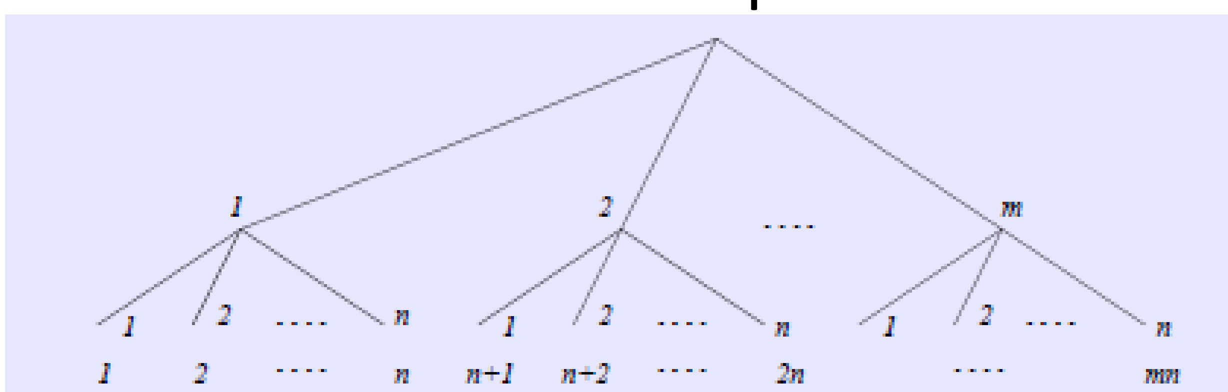
I **randomly** put the N letters into the N envelopes.

What is the probability of the event A={at least 1 person gets his letter}?

- For N = 2 letters with 2 envelopes addressed to Kashif and Javed, there are only 2 possibilities, S = {a, b} where
  - a: Kashif gets Javed's letter and Javed gets Kashif's letter,
  - b: Kashif gets Kashif's letter and Javed gets Javed's letter.
- Because of random placement, outcomes a and b are equally likely. P(A)=P({b})=1/2.
- Find P(A) when N=4.

Some Crucial Tricks

- Counting elements can be difficult. So let's consider some tricks for counting elements.
- **The multiplication principle**
  - "If a task is completed in stages (say 2 stages), so that the first stage can be completed in m ways and the second stage can be completed in n ways then the whole task can be completed in mn ways"



A fair die is rolled 3 times. Find the probability that all 3 outcomes will be different.

- S={die1 outcome x die 2 outcome x die 3 outcome}
- |S|=6*6*6=216
- A={all 3 outcomes are different}
- |A|=6*5*4=120
- P(A)=|A|/|S| = 120/216 = 0.5555

There are n people in a room, from which we need to select 3 people, one of which will serve as the president, another as the secretary and the third as the treasurer. How many ways can we make such a selection?

- President can be chosen from n people.
- That leaves n-1 people from which the secretary can be chosen.
- And finally the treasurer can be chosen from the remaining n-2 people.
- So, total number of ways that the 3 people can be chosen from n people is $n*(n-1)*(n-2)$.

Permutations

- More generally, k items can be chosen from n items in n*(n-1)*(n-2)*…*(n-(k-1)) ways.
- n*(n-1)*(n-2)*…*(n-(k-1)) = n!/(n-k)!
- **This is known as the <u>Permutations Formula</u> P(n,k)=n!/(n-k)!**
- It gives the number of ways that k items can be chosen in order from n items.

If 3 people are in a room, what is the probability that no two share the same birthday?

- Sample space S and its size |S|
- Event of interest A and its size |A|
- P(A) = |A| / |S|

Combinations

- To count the number of ways that k items can be picked "at the same time" from n items, we use the **Combinations Formula**

$$C(n,k) = n!/(k!(n-k)!)$$

- It can be understood as permutations divided by the number of repetitions

$$C(n,k) = P(n,k)/k!$$

since k items can be arranged in k! ways and all of these ways are counted as just 1 combination.

A basket contains 8 apples and 9 oranges all mixed up. We reach in, without looking, draw three items <u>all at once</u>. What is the probability that we will get 2 apples and one orange?

- Describe the sample space S and compute its size |S|.
- Describe the event of interest A and its size.
- P(A) = ?

Consider a fully shuffled standard deck of 52 playing cards. Find the probability of receiving 2 pairs while randomly drawing a hand of 5 cards.

- Sample space S and its size |S|
- Event of interest A and its size |A|
- P(A) = |A| / |S|

$$|A| = \binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$$

Probabilities via Measuring Sizes

- When the sample space
  - can not be counted, but
  - is bounded

  then probabilities can be computed by measuring sizes.

P(A)=length(A) / length(S)

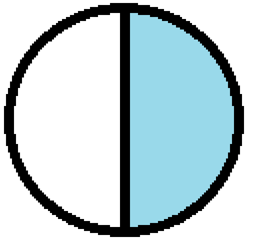
P(A)=area(A) / area(S)

P(A)=volume(A) / volume(S)

- Consider the sample space of points within a circle of radius 3 centered at the origin.

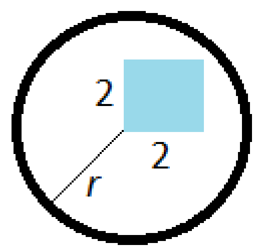
$$S = \{(x, y) : x^2 + y^2 \le 3^2\}.$$

- Can you count the number of points in S?
  - No
- Let the event A={x coordinate is >= 0}.
- A is represented by the blue area.
- S is the whole area of the circle.
- P(A) = area(A) / area(S) = 1/2



We randomly chose a point between [0,5]. What is the probability that it lies between 2 and 3.5?

Find the probability that a point chosen at random from within the circle of radius *r* lies in the blue square with side lengths 2.



S={all points in circle of radius *r*}

A={all points in the blue 2x2 square}

P(A)=?

A dart is randomly thrown at the region $S=\{(x,y): y \leq x^2, 0 < x < 4\}$ and the x-coordinate of landed spot is noted. What is the chance that the x-coordinate will lie in the interval [3,4]?

<u>Problem of Galileo</u>: Italian gamblers were puzzled as to why a sum of 10 on three rolls of a fair die seemed to occur more often than a sum of 9. Galileo wrote down the sample space and took away the mystery. Explain the mystery.