MA-250 Probability and Statistics

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Lecture 13

Independence

- Terminology:
  - **Joint probability P(A,B)**: probability of some events occurring together.
  - **Marginal probability P(A)**: probability of a single event.

Independence

Example

Let two dice (red and green) be rolled so that all the 36 possible outcomes are equally likely. Let A be the event that the red die lands 4, and let B be the event that the green die lands 4.

- S=?
- Is S simple?
- $|S|=?$
- $|A|=?$
- $|B|=?$
- $|A \cap B|=?$

Independence

$P(A) = 6/36$
$P(B) = 6/36$
$P(A)P(B) = 6/36 * 6/36 = 1/36$
$P(A \cap B) = 1/36 = P(A)P(B)$.

- Clearly, A and B are independent events. What we get on the red die has no influence on what we get on the green die.
- <u>For independent events the joint probability equals the product of marginal probabilities</u>.
- Conversely, <u>when $P(A \cap B) = P(A)P(B)$, events A and B must be independent</u>.

Independence

- There are 2 ways to look at the independence concept.
  - Intuitive
  - Probabilistic
- Intuitively, two events are independent if they do not <u>influence</u> or <u>block</u> the occurrence of each other.
- Probabilistically, two events are independent if joint probability equals the product of marginal probabilities.

Independence

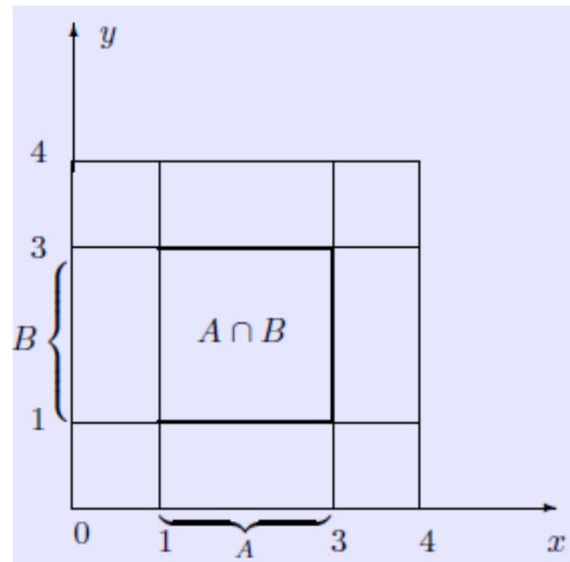
Example

Let a point be selected at random from a square, $[0, 4] \times [0, 4]$.

Let A be the event that the selected point lies in the rectangle $[1, 3] \times [0, 4]$.

Let B be the event that the selected point lies in the rectangle $[0, 4] \times [1, 3]$.

Are A and B independent?



P(A)=? P(B)=? P(A∩B)=?

Independence

Example

The sample space corresponding to the gender of 3 children of a family is as follows.

S = {BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG}.

Assume S is a simple sample space. Let A be the event that the family has both boys and girls, and let B be the event that the family has at most one girl. Are A and B independent events?

Independence

A = family has both boys and girls = {BBG, BGB, GBB, BGG, GBG, GGB},

B = family has at most one girl = {BBB, BBG, BGB, GBB}.

Therefore, $P(A) = 6/8$, $P(B) = 4/8$ and $P(A \cap B) = 3/8 = P(A)P(B)$.

H.W: Check the independence when the family has only 2 children.

Independence

Example

Let a card be drawn from a well shuffled deck of 52 cards. Let A be the event that the drawn card is an ace. Let B be the event that the drawn card is a club. Show that A and B are independent events.

Example

We select a card at random from the standard deck of 52 cards, let A be the event that a face-card is drawn (i.e., a Jack, or Queen or King is drawn). Let B be the event that the drawn card is a club. Show that A and B are independent events.

Binomial Experiments

Example

We toss a fair coin twice. Let A = {H occurs on the first toss} and let B = {H occurs on the second toss}. Are A and B independent?

The sample space is S = {HH, HT, TH, TT}. It is simple since the coin is fair.

Now A = {HH,HT}, B = {HH, TH}, $A \cap B$ = {HH}.

Therefore, $P(A) = 1/2 = P(B)$ and $P(A \cap B) = 1/4$.

A and B are independent events since $P(A \cap B) = 1/4 = 1/2 \cdot 1/2 = P(A)P(B)$.

Binomial Experiments

Example

Suppose now <u>we toss an unfair coin twice</u>, so that the probability of seeing H on a toss is *p* and it is not necessarily equal to 1/2 . How can we compute probabilities for the singleton sets of the sample space S = {HH, HT, TH, TT}?

P({HH})=p*p P({HT})=p*(1-p)

P({TT})=(1-p)*(1-p) P({TH})=(1-p)*p

Binomial Experiments

Example

Suppose now we toss an unfair coin thrice. How can we compute probabilities for the singleton sets of the sample space S?

S=? $P(\omega \in S)$=?

Binomial Experiments

Example

Now let's count the number of heads in the 3 tosses. Find

P(0 heads in 3 tosses)?

$(1-p)^3$

P(1 heads in 3 tosses)?

$p(1-p)^2$

P(2 heads in 3 tosses)?

$p^2(1-p)$

P(3 heads in 3 tosses)?

p^3

P(j heads in 3 tosses)?

$C(3,j)p^j(1-p)^{3-j}$

P(j heads in n tosses)?

$C(n,j)p^j(1-p)^{n-j}$

Terminology

Bernoulli experiment: An experiment which has only two possible outcomes is called a Bernoulli experiment. One of the outcomes is called Heads or "Success" and the other Tails or "Failure".

Bernoulli process: An experiment consisting of n unrelated (independent) and identical Bernoulli experiments.

Terminology

- **Binomial experiment:** An experiment consisting of *n* unrelated (independent) and identical Bernoulli experiments is called a binomial experiment if we count the total number of heads over the n Bernoulli trials.

Terminology

Probability of j Heads in n tosses

When a binomial experiment is performed, the probability of j Heads is

$$\binom{n}{j} p^j (1-p)^{n-j}$$

for $j = 0, 1, 2, \cdots, n$, where n is the total number of Bernoulli experiments in the binomial experiment and p is the probability of heads on one Bernoulli experiment. This is called the **Binomial model**.

Product Spaces & Independent Trials

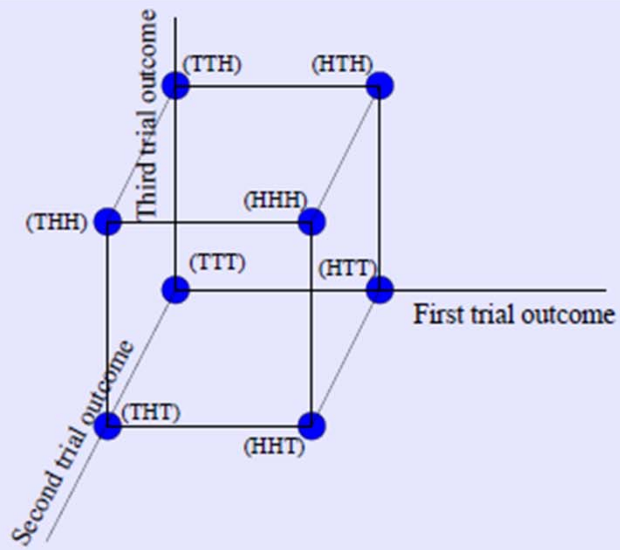


FIGURE 2.1: Sample Space as Set Product $S = S_1 \times S_2 \times S_3$.

- If a random experiment consists of *m* sub-experiments, each called a trial, the sample space, S, may be considered a <u>set product</u> of the individual sets, S_i, i = 1, 2, · · · ,m, consisting of the outcomes of each trial.
- $S = S_1 \times S_2 \times S_3 \times \cdots \times S_m$.

Product Spaces & Independent Trials

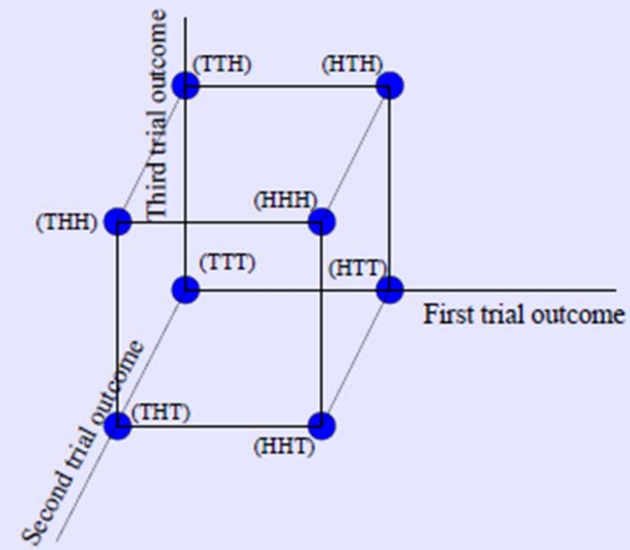


FIGURE 2.1: Sample Space as Set Product $S = S_1 \times S_2 \times S_3$.

- The resulting product set, S, becomes a product space if $P(\{\omega\}) = P(w_1)\, P(w_2) \cdots P(w_m)$ with $w_1, w_2, \cdots, w_m$ considered as outcomes over their respective trials.
- In such a case the trials are said to be independent.

“a sequence of Bernoulli trials”

“a sequence of Bernoulli trials” means that

- the trials are independent,
- each trial can have only 2 possible outcomes, (called Heads and Tails, denoted by H and T), and
- the probability of an H remains the same from trial to trial, and is usually denoted by p or r.

Binomial Experiments

For <u>a sequence of Bernoulli trials</u>, several questions of interest can be raised — resulting into several different types of random experiments.

1. Binomial experiment. Restricting over the first n Bernoulli trials, what is the probability of obtaining k number of heads?

$$\binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, 2, \cdots, n.$

Geometric experiment

2. Geometric experiment. Over the entire (infinite) sequence of Bernoulli trials, what is the probability that k tails will be observed to get the first head?

$p(1 - p)^k, \quad \text{for } k = 0, 1, 2, \cdots.$

Note the difference between the Binomial model and the Geometric model.

Shifted geometric experiment

3. Shifted geometric experiment. Over the entire (infinite) sequence of Bernoulli trials, what is the probability that k trials will be needed to get the first head?

$p(1 - p)^{k-1}$, for $k = 1, 2, 3, \cdots$.

The justification is quite similar to the one used in the geometric experiment. The difference being that now we count the trial that results into the first head.

Negative binomial experiment

4. Negative binomial experiment. Over the entire (infinite) sequence of Bernoulli trials, what is the probability that k tails will be observed to get the first 10 heads?

$$\binom{k+10-1}{10-1} p^{10}(1-p)^k \qquad \text{for } k = 0, 1, 2, \cdots$$

H.W: See if you can justify this result.

Shifted negative binomial experiment

5. Shifted negative binomial experiment. Over the entire (infinite) sequence of Bernoulli trials, what is the probability that k trials will be observed to get the first 10 heads?

$$\binom{k-1}{10-1} p^{10}(1-p)^{k-10} \qquad \text{for } k = 10, 11, 12, \cdots$$

H.W: See if you can justify this result.

Mutual Independence

Pairwise independence

If $A_1, A_2, \cdots$ are events, then this sequence is called pairwise independent if <u>every pair of two events</u> is independent.

Mutual independence

If $A_1, A_2, \cdots$ are events, then this sequence is called mutually independent if <u>every finite subset of events</u> is independent. That is

$P\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} A_j$ for every finite subset J of {1,2,3,…}.