MA-250 Probability and Statistics

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Lecture 21

Expectation of a Random Variable

- One of the central components of probability theory.
- 3 ways to look at it
  1. Long term ordinary average
  2. Centre of mass
  3. Weighted average
- For continuous R.V with density function $f_X(x)$

$$E[X] = \int_{x \in labels} x \, f_X(x)$$

- For discrete R.V with density function P(X)

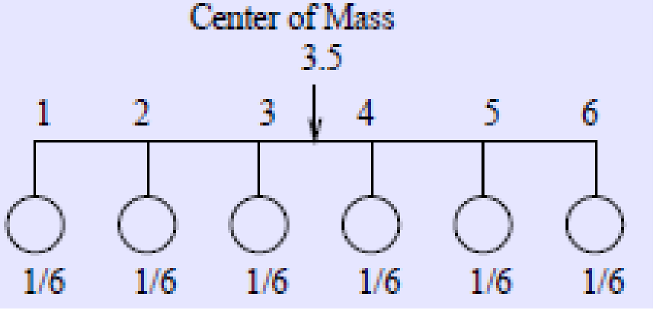
$$E[X] = \sum_{x \in labels} x \, P(X=x)$$

Expectation of a Random Variable

- In Matlab, **mean(randi([1,6], 1000000, 1))** generates 1 million integers uniformly distributed between 1 and 6 and takes their <u>average</u>.
  - Simulates average of 1 million rolls of a fair die.
- When I ran this command 10 times, my values were [3.5005 3.5034 3.4973 3.5019 3.4988 3.4965 3.5029 3.4997 3.5011 3.5054]
- Expected value is what we expect this average to be.

Expectation of a Random Variable

- For the experiment on the previous slide, let X be the face value on a roll of a fair die.
- $X \in \{1, 2, 3, 4, 5, 6\}$ with each face having equal chance of occurrence.
- The resulting probability density of X is given by $P(X=x)=1/6$ for $x=1,2,\ldots,6$.
- $E[X]=1*1/6+2*1/6+3*1/6+4*1/6+5*1/6+6*1/6=3.5$
  - This is a weighted average
  - This is also the long term ordinary average that the previous Matlab command will converge to as number of trials approaches infinity.
  - This is also the centre of mass



Expectation of a Random Variable

- Note that E[X] is not necessarily one of the labels that random variable X can take.

Expectation of a Function of a Random Variable

- Quite often in real world problems, a random variable $Y=h(X)$ is a function of some other random variable X.
- If density f_X is known, how can we find E(Y)?
- $E[Y] = \Sigma_{y \in labels}\, y\, P(Y=y) = \Sigma_{x \in labels}\, h(x)\, P(X=x)$

Expectation of a Function of a Random Variable

- Let X be the number of heads in two tosses of a fair coin.
  - $X \sim$ Binomial(2, 1/2).
- Let Y = 1/(X+1).
  - That is, h(t) = 1/(t+1) is the function that transforms X into Y.
- To find E[Y] we can
  - first find the density of Y and then use the definition of expectation, OR
  - $E[Y] = \Sigma_{x \in \text{labels}}\, h(x)\, P(X=x)$

ω	HH	HT	TH	TT
$\mathbb{P}(\{\omega\})$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$X(\omega)$	2	1	1	0
$Y(\omega)$	$\frac{1}{2+1}$	$\frac{1}{1+1}$	$\frac{1}{1+1}$	$\frac{1}{0+1}$

Linearity Property of Expectation

- Addition: $E[\ h_1(X)+h_2(X)\ ] = E[h_1(X)) + E(h_2(X)]$
- Scalar Multiplication: $E[\ ah_1(X)\ ] = aE[h_1(X)]$
- Linearity: $E[\ ah_1(X)+bh_2(X)\ ] = aE[h_1(X)] + bE[h_2(X)]$

Prove all 3 properties.

Moments of a R.V

- $E[X^k]$ is called the k^{th} moment of random variable X.
- $E[X^k]=\Sigma_{x \in labels}\ x^k\ P(X=x)$

Variance and Standard Deviation

- The quantity $E[X^2]-E[X]^2$ is called the **variance** of random variable X.
  - Denoted as Var(X).
- Square root of variance is called the **standard deviation** of X.
  - Denoted as Std(X)=sqrt(Var(X)).
- **Find $E[(X-E(X))^2]$?**

Statistics of Binomial(n,p)

- **Let X~Binomial(n,p). Find E(X), Var(X) and Std(X).**

Common Densities

- Means and variances of some of the commonly used <u>discrete</u> random variables

Random Variable	$\mathbb{E}(X)$	$Var(X)$
$X \sim B(n,p)$	np	$np(1-p)$
$X \sim Poisson(\lambda)$	λ	λ
$X \sim Geometric(p)$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$
$X \sim NB(p,m)$	$\frac{m(1-p)}{p}$	$\frac{m(1-p)}{p^2}$
$X \sim Hypergeometric(G,B,n)$	$\frac{nG}{G+B}$	$\frac{nGB}{(G+B)^2}\left(1-\frac{n-1}{G+B-1}\right)$

Common Densities

- Means and variances of some of the commonly used <u>continuous</u> random variables

Random Variable	$\mathbb{E}(X)$	$Var(X)$
$X \sim Uniform(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$X \sim Exp(\lambda)$	$1/\lambda$	$1/\lambda^2$
$X \sim G(\lambda,\alpha)$	α/λ	α/λ^2
$X \sim \chi^2_{(m)}$	m	$2m$
$X \sim Beta(a,b)$	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$
$X \sim N(a,b^2)$	a	b^2