# MA-250 Probability and Statistics

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Lecture 26

# LAW OF LARGE NUMBERS

# Law of Large Numbers

- The average of a large number of observations of a random variable X converges to the expected value E(X).
- Abbreviated as LLN.

# Application of LLN

**Monte Carlo Method for Integration**

- Let required integral under curve f(x) from a to b be region A.
  - Area(A)=required integral.
- Let sample space S be a bounding rectangle with height M and width a to b.
  - Area(S)=M(b-a)
- Let random variable X be 1 when a randomly chosen point lies in region A and 0 otherwise.
  - P(X=1)=P(A)=area(A)/area(S)
  - P(X=0)=1-P(A)

# Application of LLN

**Monte Carlo Method for Integration**

- $E(X)=0*P(X=0) + 1*P(X=1) = P(X=1) = P(A)$
- But we can estimate E(X) via LLN.
  – Randomly choose a large number of points from S and take the average of the corresponding values of random variable X.
- So $area(A)=E(X)*area(S) \approx ((x_1+x_2+\ldots+x_N)/N) / (M*(b-a))$

# CENTRAL LIMIT THEOREM

# Independent and Identically Distributed

- A set of random variables $X_1,\ldots,X_N$ is
  - **Independent** if no random variable has any influence on any other random variable.
    - $P(X_1 X_2 \ldots X_N)=P(X_1)P(X_2)\ldots P(X_N)$
  - **Identically distributed** if all random variables have the same probability density function.
    - $f_{Xi}(x)=f_{Xj}(x)$
    - Probability that random variable $X_i$ takes a value $x$ is the same as the probability that random variable $X_j$ takes the value $x$.
- Abbreviated as i.i.d variables.

# Central Limit Theorem

- Natural phenomenon can be treated as random.
- Many of them can be treated as sums of other random phenomenon.
  - $Y=X_1+X_2+\ldots+X_N$
- The Central Limit Theorem (CLT) gives us a way of finding the probability density of a sum of random variables *Y*.

# Central Limit Theorem

- Let $X_1,\ldots,X_N$ a set of i.i.d random variables with mean $\mu$ and variance $\sigma^2$.
- Let $Y=X_1+X_2+\ldots+X_N$
- **Central Limit Theorem (CLT)**:

  As $N\rightarrow\infty$, probability density of the sum $Y$ approaches a normal distribution with mean $N\mu$ and variance $N\sigma^2$.
- Notice that the $X_i$s could have any common distribution, yet the distribution of Y will converge to the normal distribution.

# Central Limit Theorem

**Equivalent formulations of CLT**:

1. As $N \to \infty$, probability density of the sum $Y$ approaches a normal distribution with mean $N\mu$ and variance $N\sigma^2$.
2. As $N \to \infty$, probability density of the average $Y_{avg}$ approaches a normal distribution with mean $\mu$ and variance $\sigma^2/N$. **(H.W: Derive this from 1.)**
3. As $N \to \infty$, probability density of the $(Y_{avg} - \mu)/(\sigma/N^{1/2})$ approaches the standard normal distribution (mean $0$ and variance $1$). **(H.W: Derive this from 2.)**

# Central Limit Theorem

**How large should N be?**

- General agreement among statisticians that N>=50 is good enough for most purposes.

# Central Limit Theorem

**Normal approximation of other distributions**

- If X~Binomial(n,p), for large values of n, the random variable (X-np)/sqrt(np(1-p)) follows N(0,1).
- Similarly for other distributions
  - Poisson
  - Uniform
  - etc.

# Central Limit Theorem

- **Find the probability of getting between 8 to 12 heads in 20 tosses of a fair coin.**

# Continuity Correction

- When approximating a discrete distribution with the Normal distribution (which is continuous), it is useful to perform a **continuity correction**.
  - $P(a \leq X \leq b) \approx P(a-0.5 \leq X' \leq b+0.5)$

# Central Limit Theorem

- **Find the probability of getting between 8 to 12 heads in 20 tosses of a fair coin. Use continuity correction.**

# Products of Random Variables

- **Expectation**: If $X$ and $Y$ are independent, then $E(XY)=E(X)E(Y)$.
- **Covariance**: $Cov(X,Y)=E[(X-\mu_X)(Y-\mu_Y)]$
- **Correlation coefficient**: $\rho(X,Y)=E[(X-\mu_X)/\sigma_X(Y-\mu_Y)/\sigma_Y]$
- Alternatively,
  - $Cov(X,Y)=E(XY)-\mu_X\mu_Y$
  - $\rho(X,Y)=Cov(X,Y)/\sigma_X\sigma_Y$