MA-110 Linear Algebra

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11. Inner Product Spaces

Inner Product

We used the dot product of vectors in $\mathbb{R}^n$ to define notions of

- length,
- angle,
- distance, and
- orthogonality.

Now we generalize those ideas to any vector space, not just $\mathbb{R}^n$.

Inner Product

An inner product on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors in V in such a way that the following 4 axioms are satisfied for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ **[Symmetry]**
2. $\langle \mathbf{u} + \mathbf{v}, w \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ **[Additivity]**
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ **[Homogeneity]**
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ **[Positivity]**

A real vector space with an inner product is called a *real inner product space*.

Inner Product
Standard

- Inner product of two vectors **u** and **v** in $\mathbb{R}^n$ was earlier defined using the dot product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- This is commonly known as the *Euclidean inner product* or *standard inner product*.
- Inner product can be defined in other ways as well – as long as the defined function satisfies the 4 axioms in the last slide.

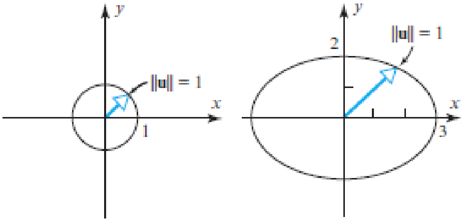
Weighted Euclidean Inner Product

- Defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

  with weights $w_1, w_2, \ldots, w_n$.
- Setting all weights to 1 yields the standard Euclidean inner product.

Weighted Euclidean Inner Product



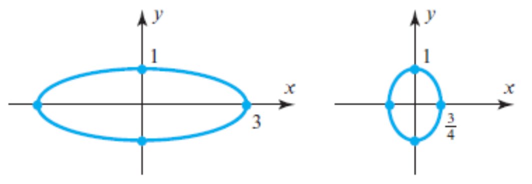
(*a*) The unit circle using the standard Euclidean inner product.

(*b*) The unit circle using a weighted Euclidean inner product.

- Left figure: Set of points at distance 1 from origin w.r.t standard Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2$.
- Right figure: Set of points at distance 1 from origin w.r.t weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{9} u_1 v_1 + \frac{1}{4} u_2 v_2$.

Weighted Euclidean Inner Product

- Sketch the unit circle in $\mathbb{R}^2$ w.r.t weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{25} u_1 v_1 + \frac{1}{49} u_2 v_2$.
- Find weighted Euclidean inner products on $\mathbb{R}^2$ for which the "unit circles" are the ellipses shown in the following figures.



Matrix inner product

- Defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v} = (A\mathbf{u})^T A\mathbf{v} = \mathbf{u}^T A^T A\mathbf{v}$$

- Also called the *inner product on $\mathbb{R}^n$ generated by A*.
- Setting $A = I$ yields the standard Euclidean inner product.
- Setting A as a diagonal matrix yields the weighted Euclidean inner product. Find A for $\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$.
- Can be viewed as standard inner product but after transforming by A.
- Plays a big role in Machine Learning, Image Processing, and Computer Vision.

Angles & Orthogonality
In General inner product spaces

- We have already seen that angle between two vectors in $\mathbb{R}^n$ can be computed using the dot product as

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}\right)$$

- Recall that dot product is a specialized form of inner product which is more general.
- *Angle between two vectors* in a *general inner product space* can be computed using the inner product as

$$\theta = \cos^{-1}\left(\frac{\langle\mathbf{u},\mathbf{v}\rangle}{||\mathbf{u}||||\mathbf{v}||}\right)$$

Angles & Orthogonality
In General inner product spaces

- Recall that $-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||} \leq 1$.
- For general inner products $-1 \leq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{u}|| ||\mathbf{v}||} \leq 1$ also holds.
- *Norm (or length)* is defined by $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.
- *Distance* between two vectors becomes $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$.
- Properties of length and distance also carry over in general spaces.
  - $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ (*Triangle inequality for vectors*)
  - $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (*Triangle inequality for distances*)
- $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ implies *orthogonality*.
  - Note that orthogonality depends on the definition of the inner product.
  - Compute $\langle \mathbf{u}, \mathbf{v} \rangle$ for $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (1, -1)$ using standard and weighted Euclidean inner product definitions.

Example
Angle between square matrices

- We have seen that matrices satisfy the 10 axioms of vector spaces.
- For $n \times n$ matrices, an inner product can be defined as $\langle \mathbf{u}, \mathbf{v} \rangle = \text{trace}(U^T V) = u_{11}v_{11} + u_{22}v_{22} + \cdots + u_{nn}v_{nn}$.
- Find the cosine of the angle between the vectors

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

- This gives us a method for computing similarities between objects in general vector spaces. *Prerequisite*: inner product needs to be defined first.