MA-310 Linear Algebra

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3. Matrix Inverse

Elementary Matrices

- Recall the 3 elementary row operations: scale, swap, add.
- If A converts into B via a sequence of elementary row operations, then B can also be converted back into A via the *inverse sequence* of elementary row operations.
- A and B are said to be *row equivalent*.
- E is called an *elementary matrix* if it can be obtained from I via a *single* elementary row operation.

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply the second row of I_2 by -3.

Interchange the second and fourth rows of I_4.

Add 3 times the third row of I_3 to the first row.

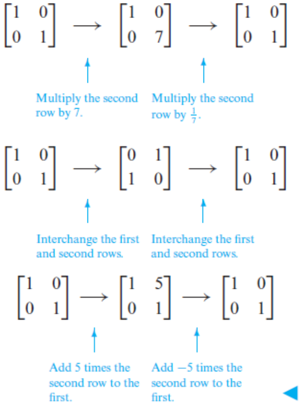
Multiply the first row of I_3 by 1.

Elementary Matrices

- $I_m \to E$ via a single elementary row operation.
- EA performs the same row operation on $A_{m\times n}$.
- Example: Through which ERO does I_2 convert to $E = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ represent? What is the effect of $E \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$?

Elementary Matrices

- For every ERO, there is an inverse ERO that recovers I.



- Every elementary matrix is invertible and the inverse is also an elementary matrix.

Equivalent Statements

- If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.
  1. A is invertible.
  2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  3. The reduced row echelon form of A is I_n.
  4. A is expressible as a product of elementary matrices.
- Proofs
  - $1 \implies 2$: Let $\mathbf{x}_0$ be *any* solution. Then $A\mathbf{x}_0 = \mathbf{0}$. Assuming 1 is true $A^{-1}A\mathbf{x}_0 = A^{-1}\mathbf{0} \implies \mathbf{x}_0 = \mathbf{0}$. So any solution *must* be the trivial solution and so $1 \implies 2$.
  - $2 \implies 3$: If 2 is true the solution can *only* be written as $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. Since the solution can be *directly read out* from the RREF, it *cannot be anything other than* I_n. So $2 \implies 3$.
  - $3 \implies 4$: If 3 is true then A and I_n are row-equivalent. So $E_k \ldots E_2 E_1 A = I_n$. So $A = E_1^{-1} E_2^{-1} \ldots E_k^{-1} I_n$. So $3 \implies 4$.

Equivalent Statements

- $4 \implies 1$: If 4 is true then $A = E_1^{-1} E_2^{-1} \dots E_k^{-1} I_n$. Since every E_i^{-1} is invertible, their sequence is also invertible and A is equal to that sequence. Hence A is invertible.

- These proofs give us a method for finding the inverse of a square matrix.
- Since $E_k \dots E_2 E_1 A = I_n$, we can right-multiply both sides by A^{-1} to obtain $E_k \dots E_2 E_1 I_n = A^{-1}$.

> The same sequence of row operations that reduces A to I_n will transform I_n to A^{-1}.

A method for finding A^{-1}

- To obtain A^{-1}, first adjoin I_n to the right side of A. That is, form the partitioned matrix $[A|I_n]$.
- Then reduce A to I_n on the left via sequence of EROs while applying the same to I_n on the right.
- If A is invertible, then when A reduces to I_n, I_n would have reduced to A^{-1}.
- Let's verify that for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \text{ the inverse is } A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

← We added −2 times the first row to the second and −1 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

← We added 2 times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← We multiplied the third row by −1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← We added 3 times the third row to the second and −3 times the third row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← We added −2 times the second row to the first.

A method for finding A^{-1}

What if A is not invertible?

- *If A is not invertible*, then it cannot be reduced to RREF.
- Therefore, if A is not invertible, then this *algorithm will produce a zero row* and stop.
- Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Solving Linear Systems via Matrix Inversion

- If A is invertible, the linear system $A\mathbf{x} = \mathbf{b}$ can be solved as $\mathbf{x} = A^{-1}\mathbf{b}$. Proof?
- So now we have seen 3 ways of solving linear systems.
  1. Gaussian elimination + back-substitution
  2. Gauss-Jordan elimination
  3. Matrix inversion (*only for square, invertible A*).
- Solve

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

Equivalent Statements

- If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.
  1. A is invertible.
  2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  3. The reduced row echelon form of A is I_n.
  4. A is expressible as a product of elementary matrices.
  5. $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ vector $\mathbf{b}$. The solution is $\mathbf{x} = A^{-1}\mathbf{b}$.
- Proof: $1 \iff 5$
  - If 1 is true then A^{-1} exists. So we can rewrite 5 as $A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$ and therefore $\mathbf{x} = A^{-1}\mathbf{b}$. So $1 \implies 5$.

Equivalent Statements

- If 5 is true then a solution to $A\mathbf{x} = \mathbf{b}$ exists *for every* $\mathbf{b}$. If a solution exists for *every* $\mathbf{b}$, then solutions exist for the following $\mathbf{b}$ vectors too.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{b}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Let those solutions be $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ and let $C = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n \end{bmatrix}$. Clearly, $\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_n \end{bmatrix} = I_n$. So $AC = I_n$ and therefore $C = A^{-1}$. So 5 $\implies$ 1.

- Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

Questions

- Exercise 1.4
  - $9, 10, 15 - 20, 23, 24, 31, 32, 34 - 36, 39, 41, 43, 46$, all true-false questions.