MA-310 Linear Algebra

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4. Diagonal and Triangular Matrices

Diagonal Matrices

Non-zero entries only on the main diagonal. Zero everywhere else.

$$D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix}, D^{-1} = \begin{bmatrix} 1/d_{11} & 0 & \dots & 0 \\ 0 & 1/d_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1/d_{nn} \end{bmatrix}$$
$$D^{k} = \begin{bmatrix} d_{11}^{k} & 0 & \dots & 0 \\ 0 & d_{22}^{k} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_{nn}^{k} \end{bmatrix}, D^{-k} = \begin{bmatrix} 1/d_{11}^{k} & 0 & \dots & 0 \\ 0 & 1/d_{22}^{k} & \dots & 0 \\ 0 & 1/d_{22}^{k} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1/d_{nn}^{k} \end{bmatrix}$$

Diagonal Matrices Multiplication

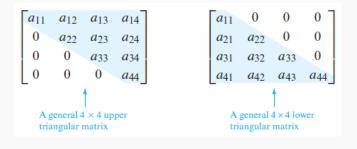
Left multiplication by a diagonal matrix

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_{11}a_{11} & d_{11}a_{12} & d_{11}a_{13} & d_{11}a_{14} \\ d_{22}a_{21} & d_{22}a_{22} & d_{22}a_{23} & d_{22}a_{24} \\ d_{33}a_{31} & d_{33}a_{32} & d_{33}a_{33} & d_{33}a_{34} \end{bmatrix}$$
$$= \begin{bmatrix} d_{11}\mathbf{r}_1 \\ d_{22}\mathbf{r}_2 \\ d_{33}\mathbf{r}_3 \end{bmatrix}$$

Right multiplication by a diagonal matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} = \begin{bmatrix} d_{11}a_{11} & d_{22}a_{12} & d_{33}a_{13} \\ d_{11}a_{21} & d_{22}a_{22} & d_{33}a_{23} \\ d_{11}a_{31} & d_{22}a_{32} & d_{33}a_{33} \\ d_{11}a_{41} & d_{22}a_{42} & d_{33}a_{43} \end{bmatrix} = \begin{bmatrix} d_{11}c_{1} & d_{22}c_{2} & d_{33}c_{3} \\ d_{11}a_{41} & d_{22}a_{42} & d_{33}a_{43} \end{bmatrix}$$

Triangular Matrices



$$a_{ij} = 0$$
 for $i > j$ $a_{ij} = 0$ for $i < j$

- Taking transpose converts lower to upper and vice versa.
- ► A triangular matrix is invertible *if and only if* its diagonal entries are all nonzero.

Triangular Matrices

- Inverse of invertible lower triangular matrix is lower triangular.
- Inverse of invertible upper triangular matrix is upper triangular.
- Product of lower triangular matrices is lower triangular.
- Product of upper triangular matrices is upper triangular.

Symmetric Matrices

A square matrix is *symmetric* if $A = A^T$. That is $a_{ij} = a_{ji}$ for all values of *i* and *j*.

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

If A and B are symmetric matrices with the same size, and if k is any scalar, then

- **1.** A^T is symmetric.
- **2.** A + B and A B are symmetric.
- 3. *kA* is symmetric.
- **4.** AB is not always symmetric. AB is symmetric if and only if A and B commute (AB = BA).
- **5**. If A is invertible, the inverse A^{-1} is also symmetric.

AA^T and A^TA

- ► Matrix products of the form AA^T and A^TA arise in a variety of applications.
- It is useful to get familiar with their properties.
- Let A be an $m \times n$ matrix. Then A^T is $n \times m$ and
 - AA^T is square $m \times m$.
 - $A^T A$ is square $n \times n$.
- Both products are symmetric since

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

$$(A^T A)^T = A^T (A^T)^T = A^T A$$

- $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$. Verify that AA^T and A^TA are symmetric.
- ► If a square matrix A is invertible, then AA^T and A^TA are also invertible.