MA-310 Linear Algebra

Nazar Khan

PUCIT

4. Diagonal and Triangular Matrices

Diagonal Matrices

Non-zero entries *only* on the main diagonal. Zero everywhere else.

$$D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix}, D^{-1} = \begin{bmatrix} 1/d_{11} & 0 & \dots & 0 \\ 0 & 1/d_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1/d_{nn} \end{bmatrix}$$

$$D^k = \begin{bmatrix} d_{11}^k & 0 & \dots & 0 \\ 0 & d_{22}^k & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_{nn}^k \end{bmatrix}, D^{-k} = \begin{bmatrix} 1/d_{11}^k & 0 & \dots & 0 \\ 0 & 1/d_{22}^k & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1/d_{nn}^k \end{bmatrix}$$

Diagonal Matrices
Multiplication

Left multiplication by a diagonal matrix

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_{11}a_{11} & d_{11}a_{12} & d_{11}a_{13} & d_{11}a_{14} \\ d_{22}a_{21} & d_{22}a_{22} & d_{22}a_{23} & d_{22}a_{24} \\ d_{33}a_{31} & d_{33}a_{32} & d_{33}a_{33} & d_{33}a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} d_{11}\mathbf{r}_1 \\ d_{22}\mathbf{r}_2 \\ d_{33}\mathbf{r}_3 \end{bmatrix}$$

Right multiplication by a diagonal matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} = \begin{bmatrix} d_{11}a_{11} & d_{22}a_{12} & d_{33}a_{13} \\ d_{11}a_{21} & d_{22}a_{22} & d_{33}a_{23} \\ d_{11}a_{31} & d_{22}a_{32} & d_{33}a_{33} \\ d_{11}a_{41} & d_{22}a_{42} & d_{33}a_{43} \end{bmatrix}$$

$$= \begin{bmatrix} d_{11}\mathbf{c}_1 & d_{22}\mathbf{c}_2 & d_{33}\mathbf{c}_3 \end{bmatrix}$$

Triangular Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general 4×4 upper triangular matrix

A general 4×4 lower triangular matrix

$a_{ij} = 0$ for $i > j$	$a_{ij} = 0$ for $i < j$

- Taking transpose converts lower to upper and vice versa.
- A triangular matrix is invertible *if and only if* its diagonal entries are all nonzero.

Triangular Matrices

- Inverse of invertible lower triangular matrix is lower triangular.
- Inverse of invertible upper triangular matrix is upper triangular.
- Product of lower triangular matrices is lower triangular.
- Product of upper triangular matrices is upper triangular.

Symmetric Matrices

A square matrix is *symmetric* if $A = A^T$. That is $a_{ij} = a_{ji}$ for all values of i and j.

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

If A and B are symmetric matrices with the same size, and if k is any scalar, then

1. A^T is symmetric.
2. $A + B$ and $A - B$ are symmetric.
3. kA is symmetric.
4. AB is *not always* symmetric. AB is symmetric if and only if A and B commute ($AB = BA$).
5. If A is invertible, the inverse A^{-1} is also symmetric.

AA^T and A^TA

- Matrix products of the form AA^T and A^TA arise in a variety of applications.
- It is useful to get familiar with their properties.
- Let A be an $m \times n$ matrix. Then A^T is $n \times m$ and
  - AA^T is square $m \times m$.
  - A^TA is square $n \times n$.
- Both products are symmetric since
  - $(AA^T)^T = (A^T)^T A^T = AA^T$
  - $(A^TA)^T = A^T(A^T)^T = A^TA$
- $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$. Verify that AA^T and A^TA are symmetric.
- If a square matrix A is invertible, then AA^T and A^TA are also invertible.