# MA-310 Linear Algebra

**Nazar Khan**

PUCIT

7. Vector Spaces

# Vectors

- Vectors in $\mathbb{R}^n$ are $n-$tuples. Ordered sets of $n$ numbers.
- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ are called *geometric vectors*.
- They can be generalized to vectors in $\mathbb{R}^n$.
- Applications of vectors
  - Digital color images $(x, y, r, g, b)$.
  - Experimental measurements.
  - Electrical circuits.
  - ... practically anything can be modelled using vectors.

# Operations on vectors

If $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, and if $k$ and $m$ are scalars, then:

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
2. $(\mathbf{u}+\mathbf{v})+w=\mathbf{u}+(\mathbf{v}+w)$
3. $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$
4. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
5. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
6. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
7. $k(m\mathbf{u})=(km)\mathbf{u}$
8. $1\mathbf{u}=\mathbf{u}$

A *linear combination* of vectors can be written as

$$\mathbf{w}=k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_r\mathbf{v}_r$$

where the scalars $k_1, k_2, \ldots, k_r$ are the *coefficents* of the linear combination.

# Norm

- The *length* or *magnitude* of a vector is called its *norm*.

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

# Unit Vector

- A vector of unit norm (length=1) is called a *unit vector*.
- Useful when only direction is important.
- Any (non-zero) vector can be *normalized* to form a unit vector in the same direction

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

- Directions of coordinate axes in a rectangular coordinate system are called the *standard unit vectors*.

| | |
|---|---|
| $\mathbb{R}^2$ | $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ |
| $\mathbb{R}^3$ | $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ |
| $\mathbb{R}^n$ | $\mathbf{e}_1 = (1, 0, 0, \ldots)$, $\mathbf{e}_2 = (0, 1, 0, \ldots), \ldots, \mathbf{e}_n = (0, 0, \ldots, 1)$ |

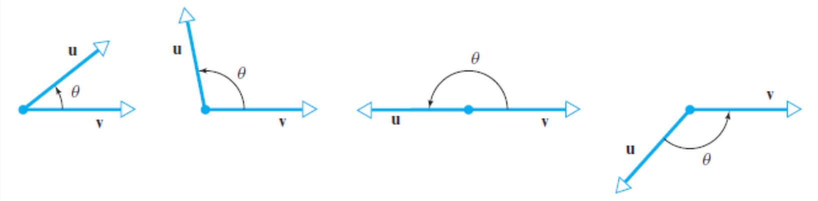
# Distance

- For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *distance* can be defined as

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

# Dot Product

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$




$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}\right)$$

Dot product enables computing angles between vectors in $\mathbb{R}^n$.

Notice that $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$. Verify this.

# Properties of Dot Product

If $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, and if $k$ is a scalar, then:

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **[Symmetry property]**
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**
3. $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ **[Homogeneity property]**
4. $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$ **[Positivity property]**

# Matrix Multiplication via Dot Products

- If the row vectors of $A$ are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ and the column vectors of $B$ are $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$, then the matrix product $AB$ can be expressed as

$$AB = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \ldots & \mathbf{r}_1 \cdot \mathbf{c}_n \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \ldots & \mathbf{r}_2 \cdot \mathbf{c}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_m \cdot \mathbf{c}_1 & \mathbf{r}_m \cdot \mathbf{c}_2 & \ldots & \mathbf{r}_m \cdot \mathbf{c}_n \end{bmatrix}$$

# Orthogonality

- In $\mathbb{R}^2$ and $\mathbb{R}^3$, vectors with an angle of $\frac{\pi}{2}$ are called *perpendicular* vectors.
- The generalization of this concept in $\mathbb{R}^n$ is *orthogonality*.

> If the angle between two vectors in $\mathbb{R}^n$ is $\frac{\pi}{2}$, they are said to be *orthogonal vectors*.

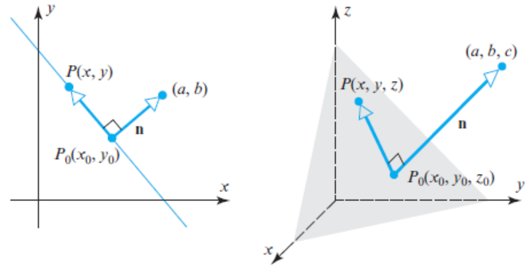
- Orthogonality is denoted by the symbol $\perp$.

> $\mathbf{u} \cdot \mathbf{v} = 0 \implies \mathbf{u} \perp \mathbf{v}$. Why?

- So the purely geometric concept of orthogonality can be captured by the purely algebraic concept of dot product.
- Are standard unit vectors in $\mathbb{R}^n$ orthogonal?

# Lines and Planes

- A line in $\mathbb{R}^2$ is determined uniquely by its slope and one of its points.
- A plane in $\mathbb{R}^3$ is determined uniquely by its inclination and one of its points.

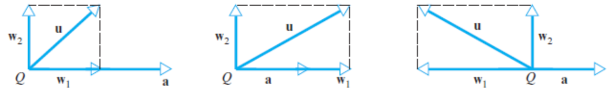



- Both can be represented algebraically as $\mathbf{n} \cdot P_0P = 0$. That is, if point $P$ lies on the line/plane, it must satisfy this equation.
- These are called the *point-normal* equations of lines/planes.

# Lines and Planes

- If $a$ and $b$ are constants that are not both zero, then an equation of the form $ax + by + c = 0$ represents a line in $\mathbb{R}^2$ with normal $\mathbf{n} = (a, b)$.
- If $a, b$, and $c$ are constants that are not all zero, then an equation of the form $ax + by + cz + d = 0$ represents a plane in $\mathbb{R}^3$ with normal $\mathbf{n} = (a, b, c)$.

# Orthogonal Projections




- Given any two vectors $\mathbf{u}$ and $\mathbf{a}$, it is always possible to decompose $\mathbf{u}$ as
$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$
where $\mathbf{w}_1$ is parallel to $\mathbf{a}$ and $\mathbf{w}_2 \perp \mathbf{a}$.
- Setting $\mathbf{w}_1 = k\mathbf{a}$, we get
$$\mathbf{u} \cdot \mathbf{a} = (k\mathbf{a} + \mathbf{w}_2) \cdot \mathbf{a} = k\mathbf{a} \cdot \mathbf{a} + (\mathbf{w}_2 \cdot \mathbf{a}) = k\|\mathbf{a}\|^2$$
- This yields $k = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}$.
- Therefore, $\mathbf{w}_1 = \text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$ and $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.

# Orthogonal Projections

- Show that $\|\mathbf{w_1}\| = \|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|} = \|\mathbf{u}\||\cos\theta|$.
- Show that for $\mathbf{u} \perp \mathbf{v}$, $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$

# Cross Product

- Only defined for $\mathbb{R}^3$.
- $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
- $\|\mathbf{u} \times \mathbf{v}\|$ represents the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.
- $\|\mathbf{u} \times \mathbf{v} \times \mathbf{w}\|$ represents the area of the parallelepiped formed by $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.