MA-110 Linear Algebra

Nazar Khan

PUCIT

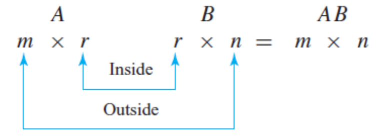
2. Matrix Arithmetic

Matrices

- Rectangular arrays of numbers with m rows and n columns.
- If $m = n$, we have a *square matrix* of order n.
- Entries $a_{11}, a_{22}, \ldots, a_{nn}$ constitute the *main diagonal*.
- *Transpose* by swapping rows and columns.
- In matrix arithmetic
  - size matters
    - $A + B$ is valid only if dimensions are equal
    - $A * B$ is valid only if dimensions match
  - order matters ($A * B \neq B * A$ generally)

Matrix Multiplication

- Multiplication is valid if columns of first matrix are equal to rows of second.



- Multiplication is carried by taking the dot-product of row i of A with column j of B.

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

$$(AB)_{ij} = a_{i1}b_{j1} + a_{i2}b_{j2} + \cdots + a_{in}b_{jn}$$

Matrix Multiplication

$$\underbrace{\begin{bmatrix} 1 & 2 & 4 \\ \color{red}{2} & \color{red}{6} & \color{red}{0} \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 4 & 1 & \color{blue}{4} & 3 \\ 0 & -1 & \color{blue}{3} & 1 \\ 2 & 7 & \color{blue}{5} & 2 \end{bmatrix}}_{3\times 4} = \underbrace{\begin{bmatrix} \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{26} & \boxed{} \end{bmatrix}}_{2\times 4}$$

$$(AB)_{23} = a_{21}b_{31} + a_{22}b_{32} + \cdots + a_{2n}b_{3n} = \color{red}{2} \cdot \color{blue}{4} + \color{red}{6} \cdot \color{blue}{3} + \color{red}{0} \cdot \color{blue}{5} = 26$$

Fill the rest.

3 ways of looking at a matrix

- Set of rows
- Set of columns
- Set of blocks (sub-matrices)

$$\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right], \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right], \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]$$

- Vector-matrix multiplication can be seen as a *linear combination* of matrix rows.
- Matrix-vector multiplication can be seen as a linear combination of matrix columns.
- Matrix-matrix multiplication can be seen as column-row expansion (sum of outer-products).

Matrix Form of a Linear System

> Every linear system can be expressed in matrix vector form and vice versa.

The linear system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3
\end{aligned}
$$

can also be written as $A\mathbf{x} = \mathbf{b}$

$$
\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}}
=
\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}
$$

where A is called the *coefficient matrix*, $\mathbf{x}$ is the vector of unknowns and $\mathbf{b}$ is the vector of constants.

Trace

- There are some operations/concepts that are defined only for square matrices.
  - Trace (sum of entries on the main diagonal)
  - Determinant
  - Inverse
  - Identity

Matrix Arithmetic Properties

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A + B = B + A$ **[Commutative law for matrix addition]**

(b) $A + (B + C) = (A + B) + C$ **[Associative law for matrix addition]**

(c) $A(BC) = (AB)C$ **[Associative law for matrix multiplication]**

(d) $A(B + C) = AB + AC$ **[Left distributive law]**

(e) $(B + C)A = BA + CA$ **[Right distributive law]**

(f) $A(B - C) = AB - AC$

(g) $(B - C)A = BA - CA$

(h) $a(B + C) = aB + aC$

(i) $a(B - C) = aB - aC$

(j) $(a + b)C = aC + bC$

(k) $(a - b)C = aC - bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

Matrix Multiplication
Be Careful!

- While most matrix arithmetic follows the rules of basic scalar arithmetic, there are some important exceptions.
- There is no such thing as matrix division!

Scalar	Matrix
$ab = ba$	$AB \neq BA$
$ab = ac \implies b = c$	$AB = AC \not\Longrightarrow B = C$
$ab = 0 \implies a = 0$ and/or $b = 0$	$AB = 0 \not\Longrightarrow A = 0$ or $B = 0$
$a^{-1} = \frac{1}{a}$	$A^{-1} \neq \frac{1}{A}$
$(a+b)^2 = a^2 + 2ab + b^2$	$(A+B)^2 \neq A^2 + 2AB + B^2$

Identity Matrix

- Identity matrix is a square, diagonal matrix containing only 1s on the diagonal and 0s elsewhere.
- I_n denotes the $n \times n$ identity matrix.
- Plays the role that 1 plays in scalar arithmetic.
- $AI_n = A$ and $I_nA = A$.
- Reduced row-echelon form of a square $n \times n$ matrix is either I_n or contains a row of zeros.

Matrix Inverse

Content in this slide applies only to square matrices.

- If A is a *square* matrix *and if there exists* another square matrix B such that $AB = BA = I$, then A is *invertible (or non-singular)* and B is the *inverse* of A.
- $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ is an inverse of $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$. Verify it.
- Any matrix with a column (or row) of zeros is not invertible. Why?
- If A and B are invertible matrices with the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. Prove it.
- Similarly, $(A_1A_2A_3 \ldots A_n)^{-1} = A_n^{-1} \ldots A_2^{-1}A_1^{-1}$.

Powers of a matrix

Content in this slide applies only to square matrices.

- $A^0 = I$
- For any integer $n > 0$, $A^n = \underbrace{AA \dots A}_{n}$.
- Also, $A^{-n} = \underbrace{A^{-1}A^{-1} \dots A^{-1}}_{n}$.
- $A^r A^s = A^{r+s}$.
- $(A^r)^s = A^{rs}$.
- For non-singular A, kA is invertible for any nonzero scalar k, and $(kA)^{-1} = \frac{1}{k}A^{-1}$. Verify that their product yields I.

Properties of the Matrix Transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(A - B)^T = A^T - B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$
- $(A_1 A_2 A_3 \dots A_n)^T = A_n^T \dots A_2^T A_1^T$.
- $(A^T)^{-1} = (A^{-1})^T$. Verify it.

Questions

- Exercise 1.3
  - 7, 8, 11, 13, 17, 25, 27, 28, 30, 35, 36, all true-false questions.