## MA-110 Linear Algebra

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4. Diagonal and Triangular Matrices

## **Diagonal Matrices**

**Diagonal Matrices** 

Non-zero entries *only* on the main diagonal. Zero everywhere else.

$$D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix}, D^{-1} = \begin{bmatrix} 1/d_{11} & 0 & \dots & 0 \\ 0 & 1/d_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1/d_{nn} \end{bmatrix}$$
$$D^{k} = \begin{bmatrix} d_{11}^{k} & 0 & \dots & 0 \\ 0 & d_{22}^{k} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_{nn}^{k} \end{bmatrix}, D^{-k} = \begin{bmatrix} 1/d_{11}^{k} & 0 & \dots & 0 \\ 0 & 1/d_{22}^{k} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1/d_{nn}^{k} \end{bmatrix}$$

# Diagonal Matrices Multiplication

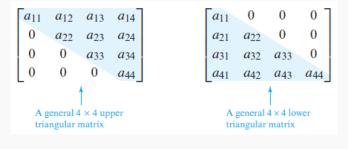
#### Left multiplication by a diagonal matrix

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_{11}a_{11} & d_{11}a_{12} & d_{11}a_{13} & d_{11}a_{14} \\ d_{22}a_{21} & d_{22}a_{22} & d_{22}a_{23} & d_{22}a_{24} \\ d_{33}a_{31} & d_{33}a_{32} & d_{33}a_{33} & d_{33}a_{34} \end{bmatrix}$$
$$= \begin{bmatrix} d_{11}\mathbf{r}_{1} \\ d_{22}\mathbf{r}_{2} \\ d_{33}\mathbf{r}_{3} \end{bmatrix}$$

Right multiplication by a diagonal matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} = \begin{bmatrix} d_{11}a_{11} & d_{22}a_{12} & d_{33}a_{13} \\ d_{11}a_{21} & d_{22}a_{22} & d_{33}a_{23} \\ d_{11}a_{31} & d_{22}a_{32} & d_{33}a_{33} \\ d_{11}a_{41} & d_{22}a_{42} & d_{33}a_{43} \end{bmatrix}$$
$$= \begin{bmatrix} d_{11}c_{1} & d_{22}c_{2} & d_{33}c_{3} \end{bmatrix}$$

## Triangular Matrices



$$a_{ij} = 0$$
 for  $i > j \mid a_{ij} = 0$  for  $i < j$ 

- ► Taking transpose converts lower to upper and vice versa.
- A triangular matrix is invertible *if and only if* its diagonal entries are all nonzero.

## **Triangular Matrices**

- ▶ Inverse of invertible lower triangular matrix is lower triangular.
- Inverse of invertible upper triangular matrix is upper triangular.
- Product of lower triangular matrices is lower triangular.
- Product of upper triangular matrices is upper triangular.

## Symmetric Matrices

A square matrix is *symmetric* if  $A = A^T$ . That is  $a_{ij} = a_{ji}$  for all values of i and j.

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

If A and B are symmetric matrices with the same size, and if k is any scalar, then

- 1.  $A^T$  is symmetric.
- 2. A + B and A B are symmetric.
- 3. *kA* is symmetric.
- **4.** AB is not always symmetric. AB is symmetric if and only if A and B commute (AB = BA).
- **5.** If A is invertible, the inverse  $A^{-1}$  is also symmetric.

## $AA^T$ and $A^TA$

- ► Matrix products of the form AA<sup>T</sup> and A<sup>T</sup>A arise in a variety of applications.
- It is useful to get familiar with their properties.
- ▶ Let A be an  $m \times n$  matrix. Then  $A^T$  is  $n \times m$  and
  - ►  $AA^T$  is square  $m \times m$ .
  - ▶  $A^T A$  is square  $n \times n$ .
- Both products are symmetric since

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

$$(A^T A)^T = A^T (A^T)^T = A^T A$$

- ▶  $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$ . Verify that  $AA^T$  and  $A^TA$  are symmetric.
- ▶ If a square matrix A is invertible, then AA<sup>T</sup> and A<sup>T</sup>A are also invertible.