MA-110 Linear Algebra

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5. Matrix Transformations

Matrix Transformations

- Special class of functions that arise from matrix multiplication.
- These functions are called "matrix transformations".
- Fundamental in the study of linear algebra.
- Important applications in physics, engineering, social sciences, and various branches of mathematics.

Basis Vectors

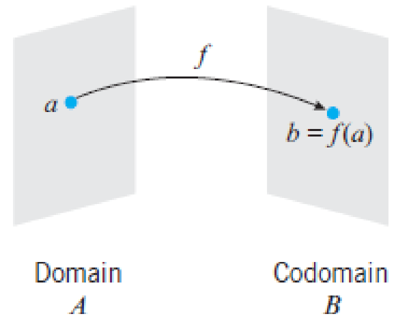
- Vectors. Default representation as a column of numbers. Denoted via lower-case, bold letters. For example, $\mathbf{x}, \mathbf{v}, \mathbf{b}$.
- Basis vectors for $\mathbb{R}^n$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

- Called basis vectors because *every* vector $\mathbf{x} \in \mathbb{R}^n$ can be represented as a linear combination of these basis vectors.

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

Functions



Usually we have considered functions as mappings from $a \in \mathbb{R}$ to $b \in \mathbb{R}$.

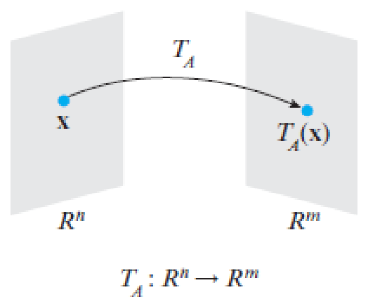
Transformations

- A function that maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ is usually called a *transformation*.
- Map vectors to vectors.
- Commonly denoted by the letter T.

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

- For $m = n$, the transformations are usually called *operators on $\mathbb{R}^n$*.
- Linear systems $A\mathbf{x}_{n\times 1} = \mathbf{b}_{m\times 1}$ can be viewed as transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Linear systems as Transformations



Represented as

- $T_A : \mathbb{R}^n \to \mathbb{R}^m$
- $\mathbf{b} = T_A(\mathbf{x})$
- $\mathbf{x} \xrightarrow{T_A} \mathbf{b}$.

Read as "T_A maps $\mathbf{x}$ onto $\mathbf{b}$".
Transformtion T_A is just "multiplication by matrix A".

Zero and Identity

- The $0_{m \times n}$ matrix containing all zeros is the *zero transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$T_0(\mathbf{u}) = \mathbf{0}$$

- I_n is the *identity operator* from $\mathbb{R}^n$ to $\mathbb{R}^n$.

$$T_{I_n}(\mathbf{u}) = \mathbf{u}$$

Properties

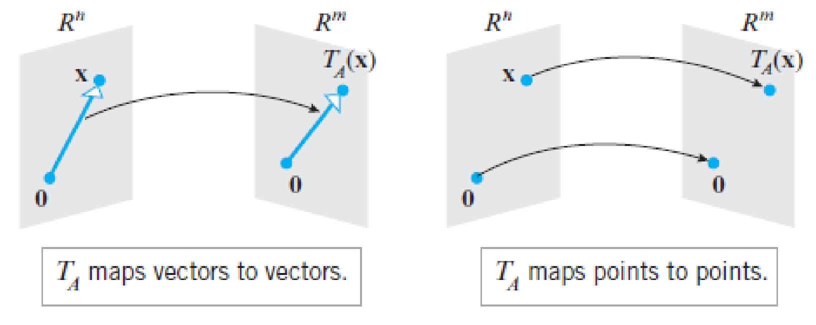
- For every matrix A the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:
  1. $T_A(\mathbf{0}) = \mathbf{0}$.
  2. $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$. *[Homogeneity property]*
  3. $T_A(\mathbf{u}+\mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$. *[Additivity property]*
  4. $T_A(\mathbf{u}-\mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$.
- Properties 2 and 3 imply *linearity*. A transformation with both properties is a *linear transformation*.
- Therefore, *matrix transformations are linear transformations*.
- It follows from 2 and 3 that

$$T_A(k_1\mathbf{u}_1+k_2\mathbf{u}_2+\cdots+k_r\mathbf{u}_r) = k_1 T_A(\mathbf{u}_1)+k_2 T_A(\mathbf{u}_2)+\cdots+k_r T_A(\mathbf{u}_r)$$

  which states that *a matrix transformation maps a linear combination of vectors in $\mathbb{R}^n$ into the corresponding linear combination of vectors in $\mathbb{R}^m$*.

Vectors vs. Points

Since n-dimensional vectors can be viewed as points in $\mathbb{R}^n$, matrix transformations can be viewed as acting on vectors or points.



Which view to take depends on the problem.

Applications of linear algebra

Polynomial Interpolation

Applications of linear algebra
Approximate Integration