MA-110 Linear Algebra

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9. Basis

Basis

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in a finite-dimensional vector space V , then S is called a basis for V if:

1. S spans V.
2. S is linearly independent.

Examples:

- Standard basis for $\mathbb{R}^n$.
- Any set of n linearly independent vectors in $\mathbb{R}^n$. (Show that the vectors $\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$.)
- Standard basis for M_{mn}.

Benefit of Basis

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in *exactly one way*.

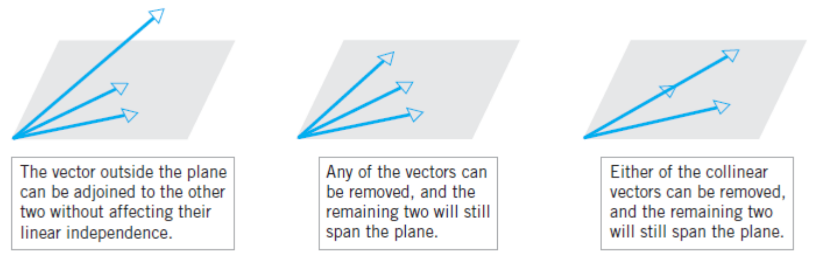
Proof: S is a basis $\implies$ $\mathbf{v}$ can be expressed in *some* way.
Assume $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ and also $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$.
Subtracting both leads to $\mathbf{0} = (c_1 - k_1)\mathbf{v}_1 + (c_2 - k_2)\mathbf{v}_2 + \cdots + (c_n - k_n)\mathbf{v}_n$.
Linear independence of S $\implies$ $(c_i - k_i) = 0$. Therefore, there can be exactly one representation of $\mathbf{v}$ in a basis.

Scalers $c_1, c_2, \ldots, c_n$ are called *coordinates* of $\mathbf{v}$ relative to basis S.
Vector $(c_1, c_2, \ldots, c_n)$ is called the *coordinate vector* of $\mathbf{v}$ relative to basis S.

Dimension

- The number of vectors in a basis for V is called the *dimension* of V.
- Denoted as $\dim(V)$.
- All basis of V must have the same dimension. Why?
- Zero vector space has dimension 0. That is $\dim(\{\mathbf{0}\}) = 0$.
- In engineering as well as computer science, dimension is sometimes referred to as *degrees of freedom*.

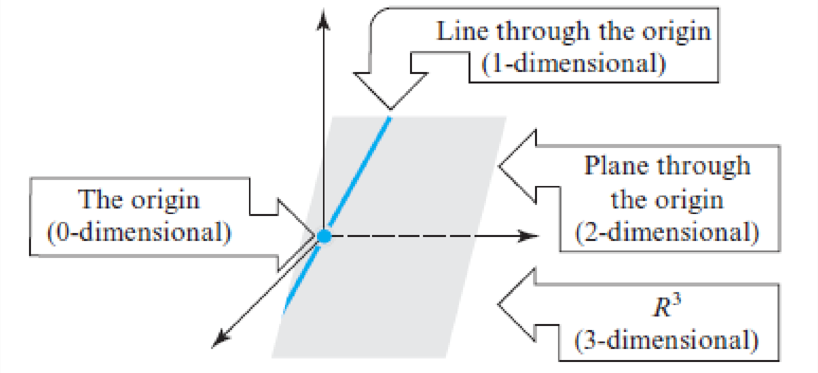
Plus/Minus Theorem



Consequences:

- If V has dimension n, then for any subset $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, it suffices to check *either* linear independence *or* spanning – the remaining condition will hold automatically.
- If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.
- If S is a linearly independent set that is not already a basis for V.

Dimension
Geometric view



Change of Basis

- Note that both $\{(1,0),(0,1)\}$ and $\{(1,1),(2,1)\}$ are valid bases for $\mathbb{R}^2$.
- But $\{(1,0),(0,1)\}$ is more convenient and commonly used.
- A basis that is suitable for one problem may not be suitable for another.
- So it is common to change from one basis to another.
- A *fixed* vector $\mathbf{v}$ in vector space V will have different coordinates relative to basis B and basis B'. Denoted by $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$ respectively.
- We will see how the new representation $[\mathbf{v}]_{B'}$ is related to the old representation $[\mathbf{v}]_B$.

Change of Basis

- Let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ be bases for $V = \mathbb{R}^2$.
- Let $\mathbf{u}'_1$ and $\mathbf{u}'_2$ be represented in the old basis B as

$$[\mathbf{u}'_1]_B = \begin{bmatrix} a \\ b \end{bmatrix} \text{ and } [\mathbf{u}'_2]_B = \begin{bmatrix} c \\ d \end{bmatrix}$$

That is

$$\begin{aligned} \mathbf{u}'_1 &= a\mathbf{u}_1 + b\mathbf{u}_2 \\ \mathbf{u}'_2 &= c\mathbf{u}_1 + d\mathbf{u}_2 \end{aligned}$$

Change of Basis

- For any vector $\mathbf{v}$ in V, let it's coordinates in basis B' be

$$[\mathbf{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

That is

$$\begin{aligned} \mathbf{v} = k_1\mathbf{u}'_1 + k_2\mathbf{u}'_2 &= k_1(a\mathbf{u}_1 + b\mathbf{u}_2) + k_2(c\mathbf{u}_1 + d\mathbf{u}_2) \\ &= (k_1 a + k_2 c)\mathbf{u}_1 + (k_1 b + k_2 d)\mathbf{u}_2 \end{aligned}$$

- Therefore, representation of $\mathbf{v}$ in the old basis B is given by

$$[\mathbf{v}]_B = \begin{bmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} [\mathbf{u}'_1]_B & [\mathbf{u}'_2]'_B \end{bmatrix} [\mathbf{v}]_{B'}$$

Change of Basis

- More generally

$$[\mathbf{v}]_B = \underbrace{\begin{bmatrix}[\mathbf{u}'_1]_B & \dots & [\mathbf{u}'_n]_B\end{bmatrix}}_{n\times n}[\mathbf{v}]_{B'} = P_{B'\to B}[\mathbf{v}]_{B'}$$

- Matrix $P_{B'\to B}[\mathbf{v}]_{B'}$ is called the *transition matrix* from basis B' to B.
- Similarly, the reverse transformation matrix is given by

$$P_{B\to B'} = \begin{bmatrix}[\mathbf{u}_1]_{B'} & \dots & [\mathbf{u}_n]_{B'}\end{bmatrix}$$

The columns of the transition matrix from an old basis to a new basis are the coordinate vectors of the old basis relative to the new basis.

Change of Basis

- Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for $\mathbb{R}^2$, where

$$\mathbf{u}_1 = (1, 0), \mathbf{u}_2 = (0, 1), \mathbf{u}'_1 = (1, 1), \mathbf{u}'_2 = (2, 1)$$

1. Find the transition matrix $P_{B' \to B}$ from B' to B.
2. Find the transition matrix $P_{B \to B'}$ from B to B'.

Change of Basis

- Since they have opposite effects

$$P_{B' \to B} = P_{B \to B'}^{-1}$$
$$P_{B \to B'} = P_{B' \to B}^{-1}$$

- A procedure for computing $P_{B \to B'}$.
  1. Form the matrix $[B'|B]$.
  2. Use elementary row operations to reduce the matrix in step 1 to reduced row echelon form.
  3. The resulting matrix will be $[I|P_{B \to B'}]$.
  4. Extract the matrix $P_{B \to B'}$ from the right side of the matrix in step 3.

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row ops.}} [\text{I} \mid \text{transition from old to new}]$$

Row Space, Column Space, and Null Space

Let A be an $m \times n$ matrix.

The subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the *row space of A*.

The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the *column space of A*.

The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of $\mathbb{R}^n$, is called the *null space of A*.

These three spaces are denoted by $\text{row}(A)$, $\text{col}(A)$, and $\text{null}(A)$ respectively.

- A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A. Why?
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Basis for row and column spaces

For matrix in row echelon form

- If a matrix R is in row echelon form, then
  - row vectors with leading 1's form a basis for the row space of R, and
  - column vectors with leading 1's of the row vectors form a basis for the column space of R.

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for row space of R

$$\mathbf{r}_1 = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \end{bmatrix}$$
$$\mathbf{r}_2 = \begin{bmatrix} 0 & 1 & 3 & 0 & 0 \end{bmatrix}$$
$$\mathbf{r}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Basis for column space of R

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{c}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Basis for row space via row reduction

- EROs do not change the row space.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \xrightarrow{EROs} R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Nonzero row vectors of R form a basis for the row space of R *and* hence form a basis for the row space of A as well.

$$\mathbf{r}_1 = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \end{bmatrix}$$
$$\mathbf{r}_2 = \begin{bmatrix} 0 & 0 & 1 & 3 & -2 & -6 \end{bmatrix}$$
$$\mathbf{r}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

Basis for column space via row reduction

- EROs change the column space.
- Column vectors of R corresponding to the leading 1's form a basis for the column space of R (<u>but not A</u>).
- The corresponding columns of A form a basis for the column space of A.

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \mathbf{c}_4 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

Rank and Nullity

Rank(A)=dim(row space of A)=dim(column space of A).

Rank(A) $\leq$ min(m, n) for $m \times n$ matrix A.

Nullity(A)=dim(null space of A)= # of free variables.

Dimension Theorem for Matrices
If A is a matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

Proof: Linear system will have n variables of 2 types – i) leading, and ii) free.