CS-565 Computer Vision

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17. Camera Anatomy

Camera Matrix

Rich Source of Geometric Information

- The 3×4 camera matrix P encodes very rich geometric information.

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}$$

- The advantage of linear algebra is that we handle all of this geometric information through algebra (manipulation of symbols).

Camera Center

- Let M denote the first 3×3 sub-matrix of the 3×4 matrix P.
- When M is non-singular, P has rank 3 and therefore a null-space of dimensionality 1.
- Therefore there exists a vector $\mathbf{v}$ such that

$$P\mathbf{v} = \mathbf{0}$$

- Vector $\mathbf{v}$ must be the camera centre $\mathbf{C}$.

Camera Center
Proof that **C** *is the null-vector of P*

- Consider the set of points along the line joining some point $\mathbf{A}$ and the camera centre $\mathbf{C}$.
$$\mathbf{X}(\lambda) = (1-\lambda)\mathbf{C} + \lambda\mathbf{A}$$
This is called the join of $\mathbf{A}$ and $\mathbf{C}$.
- All such points will map to the same image point $P\mathbf{A}$
$$\begin{aligned} \lambda P\mathbf{A} &= P\mathbf{X}(\lambda) \\ &= (1-\lambda)P\mathbf{C} + \lambda P\mathbf{A} \\ \implies P\mathbf{C} &= \mathbf{0} \end{aligned}$$
- In Python, **C**=scipy.linalg.null_space(P)

Camera Center

- Camera can image every point in $3D$ but it's own centre! Why?
- If Rank$(M) = 2$, then **C** will be a point at infinity, i.e. the last coordinate of **C** will be zero!
    - This is called the *camera at infinity* model.

Why does the road vanish at the horizon?

Why do parallel lines meet in images?

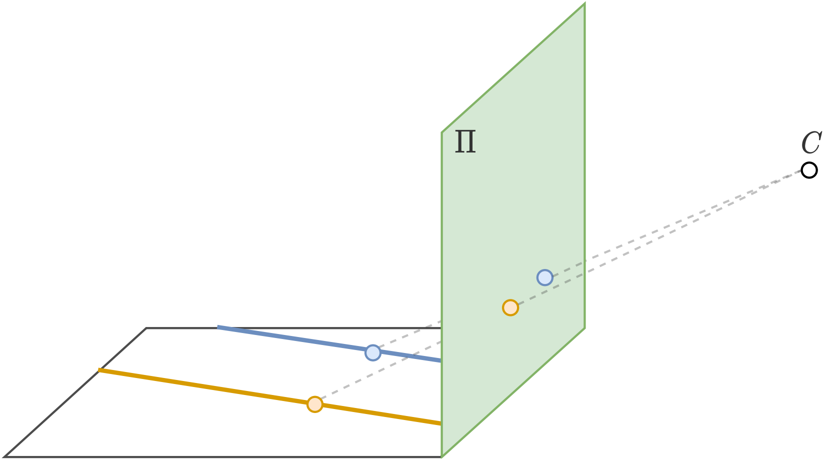


Figure: Projection of two points on two parallel lines in a plane. Author: N. Khan (2021)

Why do parallel lines meet in images?

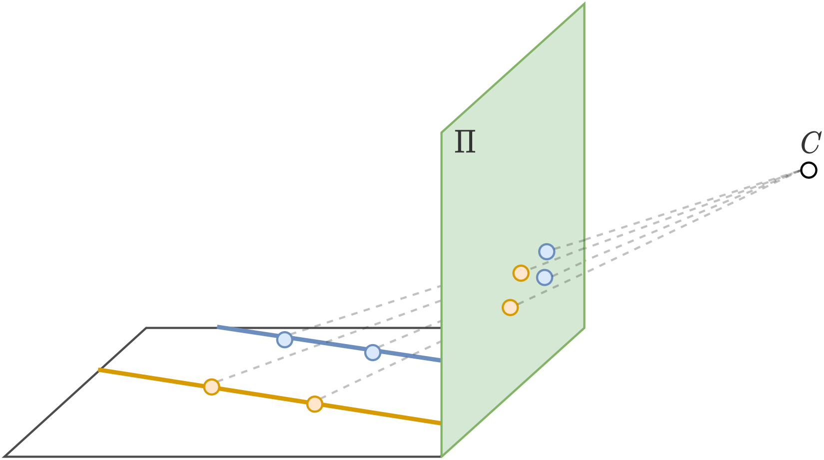


Figure: Projection of two more points. Author: N. Khan (2021)

Why do parallel lines meet in images?

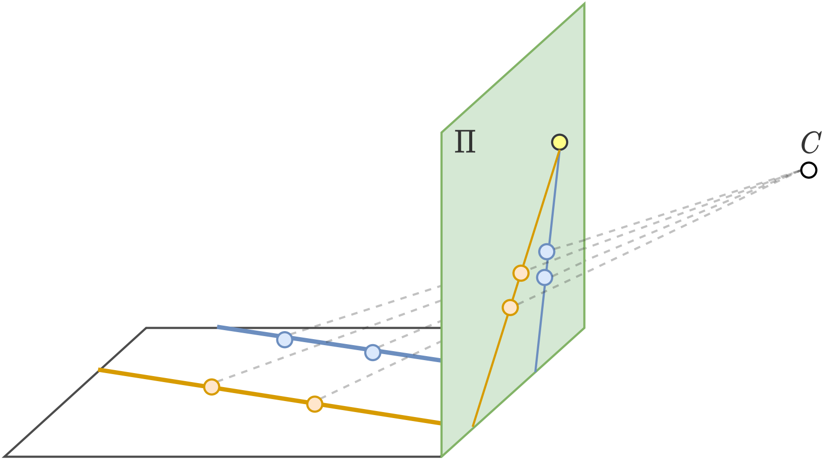


Figure: Projection of two parallel lines in a plane. Author: N. Khan (2021)

Vanishing Point

- Point where parallel lines meet in the image.
- In the real world, parallel lines meet at infinity.
- So a vanishing point is the image of infinity!

Points at Infinity

- Let $\mathbf{p}_i$ be the i-th column of P.
- Let $\mathbf{p}^{iT}$ be the i-th row of P.

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}$$

$$\mathbf{p}_i = \begin{bmatrix} p_{1i} \\ p_{2i} \\ p_{3i} \end{bmatrix} \longrightarrow P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \end{bmatrix}$$

$$\mathbf{p}^i = \begin{bmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \\ p_{i4} \end{bmatrix} \longrightarrow P = \begin{bmatrix} \mathbf{p}^{1\,T} \\ \mathbf{p}^{2\,T} \\ \mathbf{p}^{3\,T} \end{bmatrix}$$

Points at Infinity

- In homogenous coordinates we can *express points at infinity*.
  - In $\mathbb{P}^2$, $[a, b, 0]$ is a point at infinity in the direction of the $2D$ vector $[a, b]$. *Why?*
  - In $\mathbb{P}^3$, $[a, b, c, 0]$ is a point at infinity in the direction of the $3D$ vector $[a, b, c]$.
- Setting the last coordinate to 0 in homogenous coordinates, yields a point at infinity in Euclidean space.
- Every direction is represented as a point at infinity in that direction.
- Write down the representation of the x-axis in $\mathbb{P}^3$.

Columns of P

- Notice that

$$P \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T = \mathbf{p}_1 \qquad \text{(first column of } P\text{)}$$

- But $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$ is the direction of the x-axis.
- So $\mathbf{p}_1$ is the image of the point at infinity in the direction of the x-axis.
  - Also called the *vanishing point in the x-direction*.
- $\mathbf{p}_2$ is the image of ...?
- Which point at infinity maps to $\mathbf{p}_3$?
- $\mathbf{p}_4$ is the projection of ...?

Columns of P

- Column $\mathbf{p}_1$ is the vanishing point in the x-direction.
- Column $\mathbf{p}_2$ is the vanishing point in the y-direction.
- Column $\mathbf{p}_3$ is the vanishing point in the z-direction.
- Column $\mathbf{p}_4$ is the image of the world origin.

Row of P

- Each row of P contains 4 numbers that can be considered a vector in $\mathbb{P}^3$.
- Can also be considered as parameters of a plane in $3D$.
- Equation of a plane
  - Non-homogenous

$$aX + bY + cZ + d = 0$$

  - Homogenous

$$\underbrace{\begin{bmatrix} a & b & c & d \end{bmatrix}}_{\mathbf{n}^T} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = 0$$

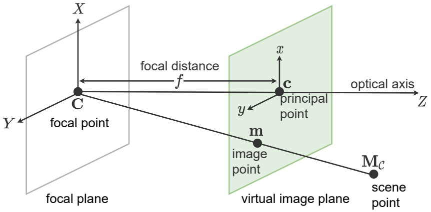
$$\mathbf{n}^T \mathbf{X} = 0$$

Rows of P

- We have seen that

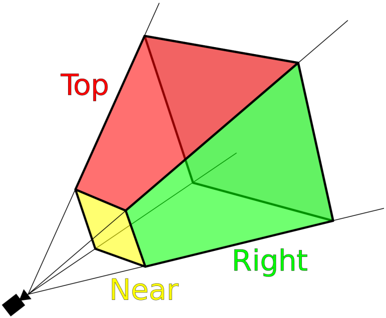
$$P = \begin{bmatrix} \mathbf{p}^{1T} \\ \mathbf{p}^{2T} \\ \mathbf{p}^{3T} \end{bmatrix}$$

- Each row $\mathbf{p}^{iT}$ is a plane in $\mathbb{P}^3$.
- All points in plane $\mathbf{p}^{3T}$ satisfy $\mathbf{p}^{3T}\mathbf{X} = 0$.
- In other words, their images are of the form $(x, y, 0)^T$.
- Therefore, $\mathbf{p}^{3T}$ *is the focal plane* since only points on that plane can have such images.



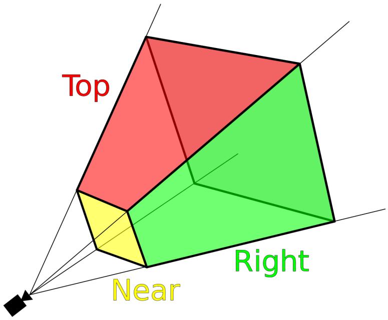
Rows of P

- All points in plane $\mathbf{p}^{1T}$ satisfy $\mathbf{p}^{1T}\mathbf{X} = 0$.
- In other words, their images are of the form $(0, y, w)^T$ which are points on the image y-axis.
- Since $P\mathbf{C} = \mathbf{0}$, $\mathbf{p}^{1T}\mathbf{C} = 0$ as well. So, $\mathbf{C}$ also lies on the plane $\mathbf{p}^{1T}$.
- Therefore, $\mathbf{p}^{1T}$ is the *plane defined by the camera centre* $\mathbf{C}$ *and the line x=0 in the image*.



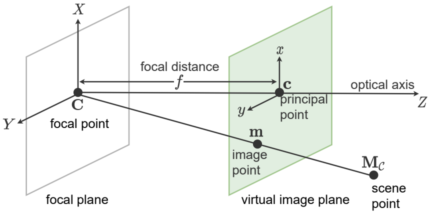
Rows of P

1. Row $\mathbf{p}^{1T}$ is the plane defined by camera centre $\mathbf{C}$ and image y-axis.
2. Row $\mathbf{p}^{2T}$ is the plane defined by camera centre $\mathbf{C}$ and image x-axis.
3. Row $\mathbf{p}^{3T}$ is the focal plane.
4. *Using 1–3 Prove that* $P\mathbf{C} = \mathbf{0}$.



Optical Axis

- Normal to plane $\begin{bmatrix} a & b & c & d \end{bmatrix}^T$ is the vector $\begin{bmatrix} a & b & c \end{bmatrix}$.
- Optical axis vector is the normal vector of the focal plane $\mathbf{P}^{3T}$.
- Therefore, it is given by $\mathbf{m}^{3T} = \begin{bmatrix} p_{31} & p_{32} & p_{33} \end{bmatrix}^T$.
- But since P is defined only upto scale, $\mathbf{m}^3$ can point in the –ve Z direction as well.
- The principal axis vector pointing to the front of the camera is given by $det(M)\mathbf{m}^3$ where M is the left 3×3 sub-matrix of P.



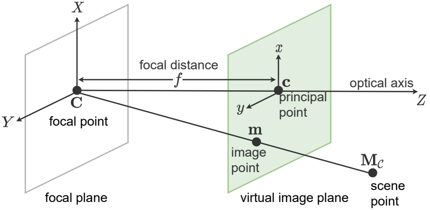
Principal Point

- Since a vector is a direction, it can be represented as $\begin{bmatrix} a & b & c & 0 \end{bmatrix}^T$ which is a point at infinity in direction $\begin{bmatrix} a & b & c \end{bmatrix}^T$.
- Optical axis vector $\mathbf{m}^3 = \begin{bmatrix} p_{31} & p_{32} & p_{33} \end{bmatrix}^T$ can be represented as a point at infinity

$$\mathbf{m}^3_\infty = \begin{bmatrix} p_{31} & p_{32} & p_{33} & 0 \end{bmatrix}^T$$

- Principal point $\mathbf{c}$ is the projection of $\mathbf{m}^3_\infty$.

$$\mathbf{c} = P\mathbf{m}^3_\infty = M\mathbf{m}^3$$



Summary

- We have seen that the camera matrix P is a rich source of geometric information.
- Given P, one can find
  - camera center $\mathbf{C}$
  - images of infinities (vanishing points)
  - image of the world origin
  - camera orientation and frustum
  - focal plane
  - principal point $\mathbf{c}$
- Next lecture: camera calibration to obtain P