CS-565 Computer Vision

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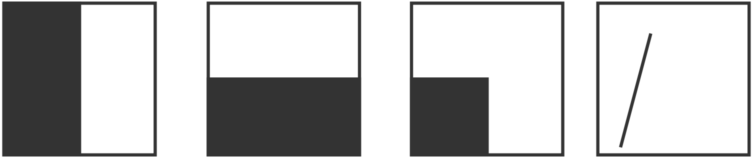
4. Approximating Derivatives

Derivatives

- Derivative represents the rate of change.
- Image derivatives represent the rate of color changes in images.
- Interesting features in images (and in the real world) have high derivatives.
- Therefore, derivatives are used for detecting semantically important features such as edges, corners and lines.



Vertical edge Horizontal edge Corner Line

Partial Derivatives and Gradient

- Let $f(x, y)$ be a 2D function.
- *Partial derivative* in x-direction is denoted by $f_x, \partial_x f$, or $\frac{\partial f}{\partial x}$.
- Higher order derivative can be computed in sequence

$$\frac{\partial^2 f}{\partial x \partial y} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

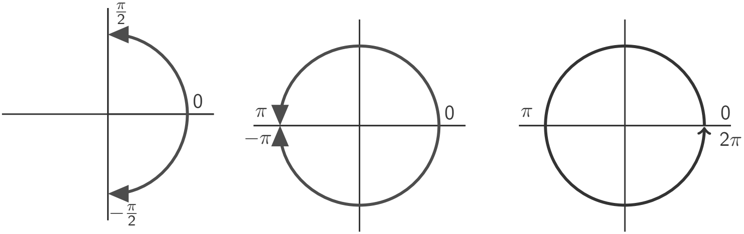
- Order of partial differentiation does not matter (under suitable smoothness assumptions)

$$f_{xy} = f_{yx}$$

- The *gradient vector* $\nabla f = (f_x, f_y)^T$ always points in the direction of highest rate of change.
- *Gradient magnitude* $|\nabla f| = \sqrt{f_x^2 + f_y^2}$ is rotationally equivariant.
- *Gradient direction* can be computed as $\theta = \arctan\left(\frac{f_y}{f_x}\right)$.

atan vs. atan2

- The atan function returns angle in the range $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ while atan2 returns angle in the range $\theta \in (-\pi, \pi)$.
  - Function atan does not differentiate between quadrants 1 and 3. Similarly for quadrants 2 and 4.
  - Function atan2 differentiates between all quadrants.
  - For example, $\text{atan}\left(\frac{1}{1}\right) = \text{atan}\left(\frac{-1}{-1}\right) = \text{atan2}(1,1) \neq \text{atan2}(-1,-1)$



$\theta = \text{atan}(\frac{y}{x})$

$\theta = \text{atan2}(y, x)$

$$\theta = \begin{cases} \text{atan2}(y,x) & \text{if } y \geq 0 \\ 2\pi + \text{atan2}(y,x) & \text{if } y < 0 \end{cases}$$

Numerical Approximation of Derivative

- Using 2nd order Taylor's expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + O(h^3) \quad (1)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) + O(h^3) \quad (2)$$

- Subtracting (2) from (1) and solving for $f'(x)$ yields

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2) \quad (3)$$

- Adding (2) and (1) and solving for $f''(x)$ gives

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h) \quad (4)$$

- Equations (3) and (4) are 1st and 2nd derivative approximations using *central differences*.

Numerical Approximation of Derivative

Take-home Quiz 2:

1. (**5 marks**) Prove that using a 1st order Taylor's expansion for $f(x+h)$ yields
$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$
This is a derivative approximation using *forward difference*.
2. (**5 marks**) Prove that using a 1st order Taylor's expansion for $f(x-h)$ yields
$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h)$$
This is a derivative approximation using *backward difference*.

Derivative approximations using central differences are more accurate since they employ 2nd order Taylor approximations. Higher than order 2 approximations will be even better but computationally more expensive.

Derivative Filters

- In images, colours are stored at every pixel. Therefore, h is (almost) always taken to be 1.
- Commonly used approximate derivative filters via convolution are

Approximation	Mask	Difference
$f'(x) = f(x+1) - f(x)$	[1, **−1**]	Forward
$f'(x) = f(x) - f(x-1)$	[**1**, −1]	Backward
$f'(x) = \frac{f(x+1) - f(x-1)}{2}$	$\frac{1}{2}$ [1, **0**, −1]	Central
$f''(x) = f(x+1) - 2f(x) + f(x-1)$	[1, **−2**, 1]	Central

where each mask will be flipped during convolution and the origin (in bold blue) will be placed over location x.

Derivative Filters

- Derivative filters are very important examples of linear shift invariant (LSI) filters.
- 1D filters can be applied on 2D images.

x-direction

$$\frac{1}{2}\begin{bmatrix} 1 \\ \mathbf{0} \\ -1 \end{bmatrix}$$

y-direction

$$\frac{1}{2}\begin{bmatrix} 1 & \mathbf{0} & -1 \end{bmatrix}$$

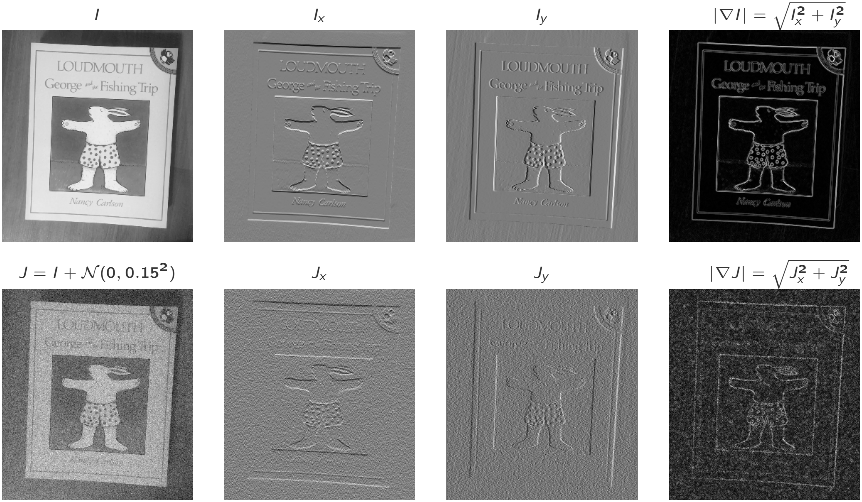
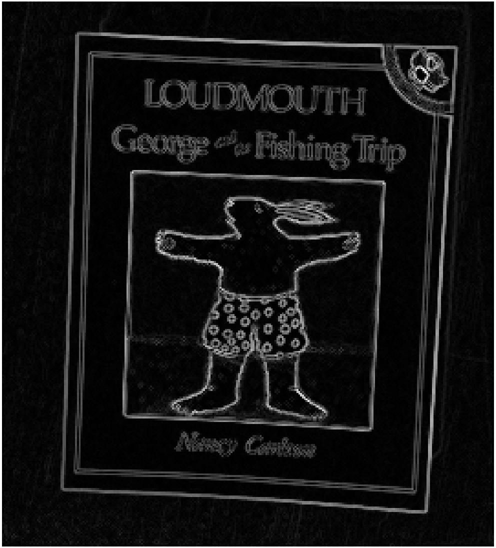


Figure: Derivative filters are sensitive to noise. Convolving with derivative filters yields high responses on edges as well as noise. Author: N. Khan (2018)


LOUDMOUTH
George and the Fishing Trip
Nancy Carlson