Cycle Discrepancy of Cubic Toeplitz Graphs

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Abstract

A Toeplitz graph is one whose adjacency matrix is a Toeplitz matrix. A Toeplitz matrix is also known as a constant diagonal matrix. This paper defines cubic Toeplitz graphs and establishes that the cycle discrepancy of a cubic Toeplitz graph is at most 1. That is $cycdisc(G) \leq 1$*, where G is a cubic Toeplitz graph. Further this bound is shown to be tight.*

Key Words: Cycle Discrepancy; Toeplitz Graphs

1. Introduction

Cycle discrepancy was introduced for the first time in [1] as a graph invariant. The inspiration of this idea is from the discipline of discrepancy theory, which is a sub-area in Combinatorics. The discrepancy theory deals with the study of deviations from the absolute uniformity or in other words, irregularities. If we are given a set *J* and a set *S* of subsets of *J*, a key question in discrepancy theory is to divide the set *J* such that each element in *S* is divided as equally as possible. If we define a labeling, $\alpha: J \rightarrow \{l_1, l_2\}$, by assigning a label to each element of *J* out of two possible labels, say l_1 and l_2, then, consequently each element in *S* will also get these labels to their elements. Note the absolute difference between number of elements having label l_1 and number of elements having label l_2 for each subset in *S*. The maximum value of absolute difference would be the discrepancy of the labeling α. For all possible such labeling we note the minimum value of discrepancy that can be achieved. The labeling which achieves the minimum is called the optimal labeling. The discrepancy of an optimal labeling is called the discrepancy of the set system (*J*, *S*). We call this quantity as discrepancy of *J* if the set *S* is understood or clear from the context. The set S, is also referred as the ground set and the set J as the collection of subsets of S. For a detailed overview of this area we refer the reader to [3, 6, 8, and 10].

A graph, $G = (V, E)$, is a pair of set of nodes and set of edges. The set, *V*, containing nodes of the graph can be taken as the ground set. A cycle in the graph can be viewed as a subset of nodes. Let us call C_G, the set containing all the cycles in the graph *G*. The discrepancy of the set system (V, C_G) is known as cycle discrepancy of the graph *G* and is denoted as $cycdisc(G)$. Due to the nature of the set system, the concept of cycle discrepancy has a graph theoretic flavor and mostly graph theoretic tools and techniques are used to investigate it.

For a cubic graph on *n* nodes, it is proved in [1] that, the cycle discrepancy is tightly bounded by $\frac{n+2}{6}$. That means if a graph (having *n* nodes) is cubic then its cycle discrepancy can be at most $\frac{n+2}{6}$, further there exists a cubic graph which achieves this bound.

The cycle discrepancy of three colorable graphs is investigated in [2] showing that cycle discrepancy of a three colorable graph is at most $2 \times \lceil \lfloor n/3 \rfloor / 2 \rceil$ and there are such three colorable graphs which achieve this bound on cycle discrepancy.

In this paper we define cubic Toeplitz graphs and show that the cycle discrepancy of a cubic Toeplitz graph is at most 1. That means if a cubic graph on *n* nodes is also a Toeplitz graph then the bound on cycle discrepancy improves from $\frac{n+2}{6}$ to 1.

The following section covers some required definitions and known results. In Section 3 cubic Toeplitz graphs are defined by means of necessary and sufficient conditions. Section 4 contains results about cycle discrepancy of cubic Toeplitz graphs. Finally Section 5 briefly concludes this work.

2. Definitions and Preliminaries

The graphs we are dealing with in this paper do not have multiple edges. They are loop less undirected graphs. We are using standard notation of graph theory mostly adopted from [5].

A symmetric square matrix X of size $n \times n$ is a Toeplitz matrix if the value $x_{i,j} = x_{0,|j-i|}$ for $0 \leq i, j \leq n-1$, hence, it is defined by only the values of the first row, as $X_n < p_0, p_1, p_2, \cdots, p_{n-1} >$, where $p_j = x_{0,j}$ for $0 \leq j \leq n-1$. It should be noted that on a diagonal all values are the same, that is why it is also known as a constant diagonal matrix. If $p_j \in \{0,1\}$ for $0 \leq j \leq n-1$, then the resulting matrix can be interpreted as an adjacency matrix of an undirected graph on n nodes. If some $p_j = 0$, we normally do not write it. In this way, to represent a Toeplitz graph on n nodes, we write $T_n < q_0, q_1, q_2, \cdots, q_k >$, for $0 < k \leq n-1$, where q_i is the diagonal containing 1's and there are at most $n-1$ such diagonals as we are avoiding self-loops.

A labeling, χ, of a graph is to map set of nodes to the set $\{+1, -1\}$. We will simply use '$+$' and '$-$' instead of $+1$ and -1. For any subset, *C*, of nodes from a given graph *G*, define,

$$\chi(C) = \sum_{u \in C} \chi(u).$$

A cycle in a graph consists of a subset of nodes. Consider a set which contains all the cycles of a graph *G*, call it C_G. The cycle discrepancy of a labeling, χ, can be defined as:

$$cycdisc(\chi) = \max_{C \in C_G} |\chi(C)|.$$

The cycle discrepancy of a graph, *G*, represented as $cycdisc(G)$, can be defined as:

$$cycdisc(G) = \min_{\chi: V \to \{+,-\}} cycdisc(\chi).$$

It is important to note that the cycle discrepancy of a bipartite graph is zero and any graph with a cycle discrepancy zero is bipartite. The cycle discrepancy of a graph is always greater than or equal to the cycle discrepancy of any of its sub graphs. Hence a graph containing an odd cycle has cycle discrepancy at least one.

For a given graph, $G = (V, E)$, If a mapping, $f: V \to \{r, g, b\}$, exists such that for any two nodes u and v which are connected, $f(u) \neq f(v)$. If a graph, G, admits such a tri-coloring then it is called a 3-colorable graph. A tri-coloring divides the set of nodes into three classes. Each class has nodes of the same color. These color classes can also be described as, $X = f^{-1}(r), Y = f^{-1}(g)$, and $Z = f^{-1}(b)$. "This tri-coloring can be characterized by permuting the colors of the partitions of set of nodes (color classes). Therefore we normally say that $f = (X, Y, Z)$ is a tri-coloring of a given graph G" [2]. We assume here that Z is the smallest color class and X is the largest color class.

We are using in this work, Lemma 5 from [1] and for the convenience of the reader it is reproduced here as Lemma 1 with few changes for clarity.

Lemma 1: [Lemma 5 [1]]

"Let $G = (V, E)$ be a K4-free, three-regular graph. Let $c = (X, Y, Z)$ be a tri-coloring of G such that each vertex z ∈ Z has at least one neighbor in X and at least one neighbor in Y. Define, $k = |X| - |Y|$ and $q = |Y| - |Z|$. Then for the labeling ρ we have,

$$cycdisc(\rho) \leq \frac{n + 2k - 2q}{6}. \quad \blacksquare$$

The Lemma 1 can be extended when $k = 0$. We record this extension as Corollary 1.

Corollary 1:

If $k = 0$, then for the labeling ρ we have,

$$cycdisc(\rho) \leq \frac{|Z|}{2}.$$

Proof:

As $n = |X| + |Y| + |Z|$, using Lemma 1 we have,

$$\begin{aligned} cycdisc(\rho) &\leq \frac{n + 2k - 2q}{6} \\ &= \frac{3|X| - 3|Y| + 3|Z|}{6} \\ &\leq \frac{||X| - |Y|| + |Z|}{2}. \end{aligned}$$

Note that if $k = 0$ then $|X| = |Y|$ hence,

$$cycdisc(\rho c) \leq \frac{|Z|}{2}. \quad \blacksquare$$

It is important to note that if we want a lower bound, *B*, on cycle discrepancy then we have to present a graph which shows the following property. For every labeling of the graph there is a cycle whose discrepancy is at least B_L. On the other hand for upper bound, B_U, a labeling has to be shown such that every cycle of the graph has discrepancy at most B_U.

3. Cubic Toeplitz Graphs

A cubic graph is one in which every node has exactly three neighbors. We have already defined Toeplitz graphs in Section 2. Here we first give an example of cubic Toeplitz graph followed by a formal definition of the same.

Recall the definition of Toeplitz graph from Section 2. As an example if we write $T_8 < 3, 4, 5 >$, it would mean an adjacency matrix of size 8×8 with diagonal number 3, 4, and 5 containing 1's and all other diagonals contain zeros. The cubic Toeplitz graph, $T_8 < 3, 4, 5 >$ having eight nodes is shown in Figure 1 drawn in three different ways showing three isomorphic versions.

Now we define three regular or cubic Toeplitz graph, $T_n < x, y, z >$, on 'n' nodes with exactly three diagonals having ones, we assume throughout the text that $x < y < z$.

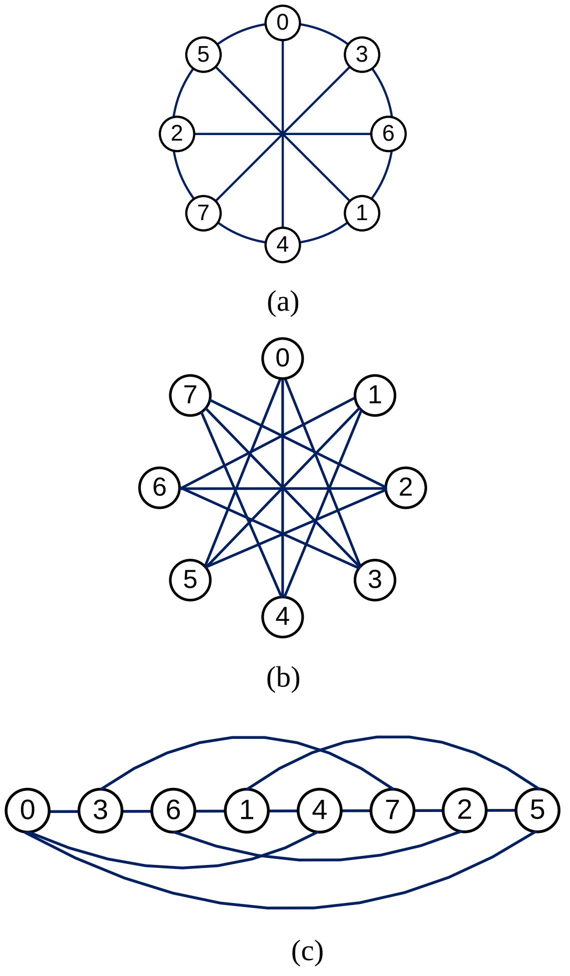

Figure 1: A cubic Toeplitz graph, $T_8 < 3, 4, 5 >$, on eight nodes. **(a)** Circular version **(b)** Star version **(c)** Path version.

Theorem 1:

An undirected Toeplitz graph $T_{n=2y} < x, y, z >$ is cubic if and only if $y = \frac{n}{2}$ and $x + z = n$.

Proof:

(a) If y is less than $\frac{n}{2}$ then x would also be less than $\frac{n}{2}$ such that $x < y < \frac{n}{2}$. The resulting degree of node number $\frac{n}{2}$ will be greater than three.

(b) If y is greater than $\frac{n}{2}$ then z is also greater than $\frac{n}{2}$ such that $\frac{n}{2} < y < z$. The resulting degree of node number $\frac{n}{2}$ will be less than three.

From (a) and (b) it is clear that if $y \neq \frac{n}{2}$ then the graph cannot be cubic. In other words, if the Toeplitz graph is cubic then $y = \frac{n}{2}$.

Note that diagonal number d contains $n - d$ ones above the main diagonal. In a cubic Toeplitz graph y has contributed $\frac{n}{2}$ ones, because $n - y = n - \frac{n}{2} = \frac{n}{2}$. In a cubic graph on n nodes the number of edges is $\frac{3n}{2}$, where, $n > 3$. After the edges contributed by y, the remaining n edges have to be contributed by diagonals x and z collectively.

(c) If $x + z > n$, then the edges contributed by x and z are: $n - x + n - z = 2n - (x + z) < n$.

(d) If $x + z < n$, then the edges contributed by x and z are: $2n - (x + z) > n$.

From (c) and (d) it is clear that if $x + z \neq n$, then the graph is not cubic. In other words, if the graph is cubic then $x + z = n$.

Conversely, now we show that if $y = \frac{n}{2} \wedge (x + z) = n$, then the resulting Toeplitz graph is cubic. Note that while moving along a diagonal, row number and column number are changing continuously. As diagonal y is at $\frac{n}{2}$, it contributes exactly one to the degree of each node by connecting node number i to $\frac{n}{2} + i$, where $0 \leq i \leq \frac{n}{2} - 1$.

Due to diagonal number x each node number i would get incremented if $i - j = x$ or $j - i = x$ for $j = 0, 1, \ldots, n - 1$. The degree of node number $0, 1, \ldots, x - 1$ and $n - x + (0), n - x + (1), \ldots, n - x + (x - 1)$ is incremented by exactly one, while the degree of node number $x, x + 1, x + 2, \ldots, n - (x + 1)$ is incremented by exactly two. Each such node numbered p, getting a degree increment of exactly two would be connected to $p + x$ and $p - x$. i.e. $(p + x) - p = x$ and $p - (p - x) = x$.

Due to diagonal number z first $n - z$ nodes are connected to last $n - z$ nodes incrementing their degree by exactly one. Note that $n - z = x$, and these are exactly those nodes which got only an increment of one in their degree by diagonal x.

Hence each node has degree exactly three by collective contribution of diagonals x, y, and z, showing that it is a cubic graph. ■

4. Cycle Discrepancy of Cubic Toeplitz Graphs

In this section we will discuss the connectivity of cubic Toeplitz graphs, followed by their cycle discrepancy.

Theorem 2:

A three regular Toeplitz graph $T_n < x, y, z >$, is connected if and only if $g.c.d(x, y) = 1$.

Proof:

Note that

$$g.c.d(x, y, z) = g.c.d(x, y),$$

Because

$$y - x = z - y.$$

Let us call $y - x = d$, then

$$g.c.d(x, y, z) = g.c.d(x, x + d, x + 2d) = g.c.d(x, d)$$

And, $g.c.d(x, y) = g.c.d(x, d)$.

As the graph is cubic, $n = 2y$. Assume that $g.c.d(x, y, z) = k$, then we can say that x, y, z and n are multiples of k.

Reachability in a graph is an equivalence relation. Every node number in set $\{px, qy, rz\}\ mod\ n$ will be reachable from node number zero where p, q and r, are positive integers. Note that px, qy and rz are multiples of k and all these nodes make an equivalence class C_0. If there are node numbers which are congruent to one mode k, they are not in class C_0 and in that case the graph is not connected. Further note that there will be k equivalence classes which are not reachable from each other.

If $g.c.d(x, y) = 1$, we know that $n = 2y$, then $g.c.d(x, n) \in \{1, 2\}$.

If $g.c.d(x, n) = 1$, then node number zero will be reachable to nodes numbered $\{x, 2x, 3x, \ldots, (n-1)x\}\ mod\ n$, as $nx \equiv 0\ mod\ n$. Hence the graph is connected.

If $g.c.d(x, n) = 2$, then there will be two equivalence classes, one with even numbered nodes and other with odd numbered nodes. Further we can observe that zero is connected to y which is odd. Hence graph is connected. ■

Theorem 3:

A connected cubic Toeplitz graph $T_n < x, y, z >$, is bipartite if $g.c.d(x, n) = 1$ and y is an odd number.

Proof:

As graph is cubic so n is even. Note that x is an odd number because $g.c.d(x, n) = 1$. We can arrange all the nodes on a cycle by using only the edges of diagonals x and z because x is relatively prime to n and $z = n - x$. As this is an even cycle, it is bipartite.

Now put the edges of diagonal y which is an odd number and hence it will connect nodes which are at an odd distance with each other on the cycle. Hence each y edge will be across partitions. ■

Theorem 4:

A connected cubic Toeplitz graph $Tn < x, y, z >$, has cycle discrepancy one if $g.c.d(x, n) = 1$ and y is an even number.

Proof:

All the nodes can be arranged on a cycle as x and n are relatively prime. Let us define a labeling ρ in the following manner. Assign alternating labels to the nodes on the cycle skipping node number y and z. Assign labels to the skipped nodes opposite to the majority of their neighbors. Node number z will get the same label as node number zero and node number y will get the label opposite to node number zero. Now let us elaborate this labeling while dividing the graph into three partitions.

Define three partitions X, Y, and Z such that X contains only positively labeled vertices and Y contains only negatively labeled vertices. The partition Z may contain vertices of both kinds. Put node number zero in X partition and node number x in the Y partition. In this way we put node number $2ix\ mod\ n$ in partition X, and $(2i + 1)x\ mod\ n$ in partition Y, for $i = 0, 1, 2, \ldots (y/2) - 1$. The next node on the cycle is node number $yx\ mod\ n$, we put it in Z and node number $yx + 1\ mod\ n$ in partition X. Now put node number $(2i + 1)x\ mod\ n$ in partition X for $i = \left(\frac{y}{2}\right) + 1, \left(\frac{y}{2}\right) + 2, \ldots, \left(\frac{y}{2}\right) + \left(\frac{y}{2} - 2\right)$ and node number $2ix\ mod\ n$ in partition Y for $i = \left(\frac{y}{2}\right) + 1, \left(\frac{y}{2}\right) + 2, \ldots, \left(\frac{y}{2}\right) + \left(\frac{y}{2} - 1\right)$. The last node numbered $(2y - 1)x\ mod\ n$ which is congruent to z is placed in partition Z.

The node numbered $(2y - 1)x\ mod\ n$ is congruent to z and is connected to node number zero which is positive. The other two neighbors of this node are $(y - 1)x\ mod\ n$ and $2(y - 1)x\ mod\ n$ both of which are negative. Hence this node is labeled positive.

The node numbered $yx\,mod\,n$ is congruent to y (because x is odd and $x-1$ is even hence $(x-1)y$ is a multiple of n so xy leaves a remainder y when divided by n) and is connected to zero which is positive. Node number y is also connected to two nodes which are adjacent to it on the cycle, one of which is positively labeled and the other is negatively labeled. Hence the node number $yx\,mod\,n$ is labeled negative.

Observe that both the nodes in Z have a neighbor in X and a neighbor in Y, further, |X| = |Y|, so we can apply Corollary 1.

$$cycdisc(\rho) \leq \frac{|Z|}{2} = 1. \quad (1)$$

Now for the other side we note that:

$0 \sim x \sim 2x \sim 3x \sim \ldots \sim (y-1)x \sim yx \sim 0\ (mod\ n)$, is an odd cycle in the graph as y is an even number. Hence

$$cycdisc(T_n) \geq 1. \quad (2)$$

Combining both Eq. 1 and Eq. 2 we conclude that:

$$cycdisc(T_n) = 1. \quad \blacksquare$$

Theorem 5:

A cubic Toeplitz graph $T_n < x, y, z >$, is either bipartite or has cycle discrepancy one.

Proof:

If the graph $T_n < x, y, z >$, is connected then $g.c.d(x,y) = 1$ and $g.c.d(x,n) \in \{1,2\}$. The case when $g.c.d(x,n) = 1$, by Theorem 3 and Theorem 4, $T_n < x, y, z >$ is either bipartite or it has cycle discrepancy one.

The case when $g.c.d(x,n) = 2$, both x and n will be even. If we consider only diagonals x and z, all even numbered nodes will be making one cycle and odd numbered nodes will be making another cycle as shown in Figure 2. The $\frac{n}{2}$ edges due to diagonal y will be joining these cycles. Next we define a labeling, ρ, to establish a bound on the cycle discrepancy.

Consider the cycle containing node number zero and label it with alternating labels starting with node number zero (positive label) and then node number x (negative label). The last node in this cycle will be node number $n-x$ which will get the same label as node number zero. Now looking at $\frac{n}{2}$ edges going across the two cycles, label the cycle containing odd numbered nodes with labels opposite to the even numbered neighbors. At this stage we have exactly two edges connecting the nodes with same label. One edge will be between node number zero and $n-x$ and the other edge will be between node number $\frac{n}{2}-x$ and $\frac{n}{2}$.

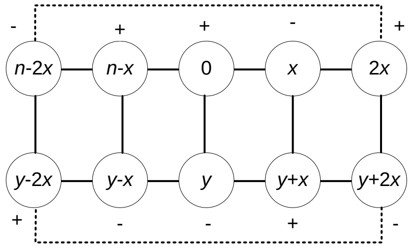

Figure 2: The case when $g.c.d(x,n) = 2$, then all even numbered nodes will be making one cycle and odd numbered nodes will be making another cycle. The node numbers should be treated modulo n.

Now let us partition the set of nodes into three sets as following.

$$P_1 = \{Nodes\ with\ positive\ label\}/P_3$$
$$P_2 = \{Nodes\ with\ negative\ label\}/P_3$$
$$P_3 = \{n-x, n/2\}$$

Note that node number zero and $n-x$ have positive label whereas node number $\frac{n}{2}-x$ and $\frac{n}{2}$ have negative label. Note $n-x$ and $\frac{n}{2}$ are not connected with each other and have different labels. Further both of these nodes have a neighbor in P_1 and a neighbor in P_2. We can apply Corollary 1 while observing that $|P_1| = |P_2|$.

$$cycdisc(\rho) \leq (|P_3|)/2 = 1. \quad (3)$$

Both the cycles created by only x and z edges have odd number of nodes because $y = \frac{n}{2}$ is odd. Hence

$$cycdisc(T_n) \geq 1. \quad (4)$$

Combining both Eq. 3 and Eq. 4 we conclude that:

$$cycdisc(T_n) = 1.$$

Lastly if the graph is not connected then

$g.c.d(x, y)$ will be greater than one. Assume that $g.c.d(x, y) = k$, where $k > 1$ then the graph will consist of k components and each component is isomorphic to $T_{\frac{n}{k}} < \frac{x}{k}, \frac{y}{k}, \frac{z}{k} >$ which is a connected cubic Toeplitz graph with $g.c.d\left(\frac{x}{k}, \frac{y}{k}\right) = 1$. Hence each component is either bipartite or has cycle discrepancy one. Hence the overall graph is either bipartite or has cycle discrepancy one. ■

The results presented in this section are summarized in Figure 3. The root node tells that the given graph is cubic Toeplitz graph and if it is not connected the argument is covered in Theorem 5. For the connected graphs, if $g.c.d(x, n) = 2$, the argument is presented in Theorem 5. Otherwise $g.c.d(x, n) = 1$ and it splits into further two cases covered in Theorem 3 and Theorem 4.

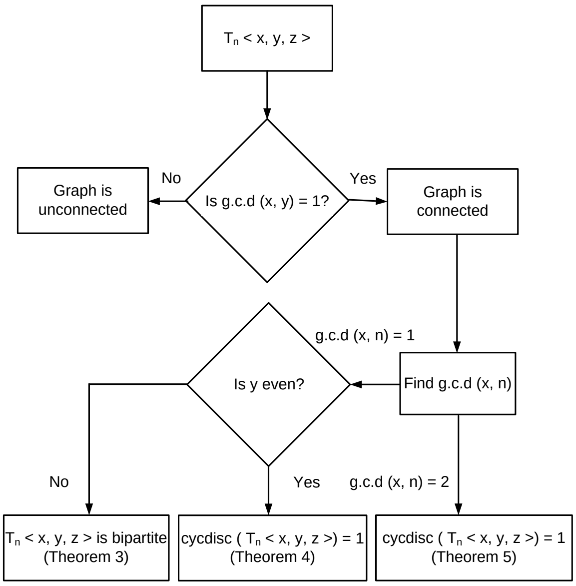

Figure 3: Completeness of argument for different cases with relevant theorem numbering is shown.

Conclusions

In this paper, cubic Toeplitz graphs are defined and further tight bound on their cycle discrepancy is presented. It is shown that the cycle discrepancy of a cubic Toeplitz graph is at most 1. Further there are cubic Toeplitz graphs containing odd cycle(s), making the stated bound tight.

At the end, we present some open problems in this area.

Problem 1: What is the bound on cycle discrepancy of a 4-regular Toeplitz graph?

Problem 2: What is the bound on cycle discrepancy of a Toeplitz graph with m nonzero diagonals in its adjacency matrix?

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